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U-Duality and the compactified Gauss-Bonnet term

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Abstract: We present the complete toroidal compactification of the Gauss-Bonnet Lagrangian from D dimensions to $D-n$ dimensions. Our goal is to investigate the resulting action from the point of view of the "U-duality" symmetry $\mathrm{SL}(n+1,\mathbb{R})$ which is present in the tree-level Lagrangian when $D-n=3$. The analysis builds upon and extends the investigation of the paper [arXiv:0706.1183], by computing in detail the full structure of the compactified Gauss-Bonnet term, including the contribution from the dilaton exponents. We analyze these exponents using the representation theory of the Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ and determine which representation seems to be the relevant one for quadratic curvature corrections. By interpreting the result of the compactification as a leading term in a large volume expansion of an $\mathrm{SL}(n+1,\mathbb{Z})$-invariant action, we conclude that the overall exponential dilaton factor should not be included in the representation structure. As a consequence, all dilaton exponents correspond to weights of $\mathrm{sl}(n+1,\mathbb{R})$, which, nevertheless, remain on the positive side of the root lattice.

Keywords: Discrete and Finite Symmetries, Global Symmetries, M-Theory, String Duality.

Contents

1. Introduction and summary

Dimensional reduction of supergravity theories is an efficient method of revealing symmetry structures which are "hidden" when the theories are formulated in maximal dimension. The first discovery of such a hidden symmetry was the so-called Ehlers symmetry of pure four-dimensional gravity compactified on a circle to three dimensions [1]. The global symmetry $\mathrm{GL}(1,\mathbb{R})=\mathbb{R}$, corresponding to rescaling of the S^1, is in this case extended through dualisation of the Kaluza-Klein vector into a new scalar, revealing that the full global symmetry of the Lagrangian is, in fact, described by the group $\mathrm{SL}(2,\mathbb{R})$. The scalars in the theory parametrise the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, where $\mathrm{SO}(2)$ is the maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$, playing the role of a local gauge symmetry. More generally, upon toroidal compactification of lowest order pure gravity in D spacetime dimensions on an n-torus, T^n, to three dimensions, the scalars parametrise the coset space $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+$

1). The enhancement from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{SL}(n+1,\mathbb{R})$ is again due to the fact that in three dimensions all Kaluza-Klein vectors can be dualised to scalars.

Similar phenomena occur also for coupled gravity-dilaton-p-form theories, such as the bosonic sectors of the low-energy effective actions of string and M-theory. The most thoroughly investigated case is the toroidal compactification of eleven-dimensional supergravity on T^n to $d = 11 - n$ dimensions, for which the scalar sector parametrises the coset space $\mathcal{E}_{n(n)}/\mathcal{K}(\mathcal{E}_{n(n)})$, with $\mathcal{K}(\mathcal{E}_{n(n)})$ being the (locally realized) maximal compact subgroup of $\mathcal{E}_{n(n)}$ [2]. In particular, for reduction to three dimensions the global symmetry group is the split real form $\mathcal{E}_{8(8)}$, with maximal compact subgroup $\mathrm{Spin}(16)/\mathbb{Z}_2$. The global symmetry group $\mathcal{E}_{8(8)}$ is the U-duality group, which, from a string theory perspective, combines the non-perturbative S-duality group $\mathrm{SL}(2,\mathbb{R})$ of type IIB supergravity with the perturbative T-duality group $\mathrm{SO}(7,7)$ [3].

These symmetries are present in the classical (tree-level) Lagrangian, but it is known from string theory that they must be broken by quantum effects. It has been conjectured that if $\mathcal{U}_d$ is the continuous symmetry group appearing upon compactification from D to $d = D - n$ dimensions, then a discrete subgroup $\mathcal{U}_d(\mathbb{Z}) \subset \mathcal{U}_d$ lifts to a symmetry of the full quantum theory [4−6].[1] The physical degrees of freedom of the scalar sector then parametrise the coset space $\mathcal{U}_d(\mathbb{Z})\backslash\mathcal{U}_d/\mathcal{K}(\mathcal{U}_d)$.

1.1 Non-Perturbative completion and automorphic forms

Recently, several authors [7−11] have initiated an investigation aimed at answering the question of whether or not the U-duality group $\mathcal{U}_3$ in three dimensions is preserved also if the tree-level Lagrangian is supplemented by higher order curvature corrections. The consensus has been that toroidal compactifications of quadratic and higher order corrections give rise to terms which are not $\mathcal{U}_3$-invariant.[2]

A nice example of a fairly well understood realisation of these mechanisms is the breaking of the classical $\mathrm{SL}(2,\mathbb{R})$ symmetry of the type IIB supergravity effective action down to the quantum S-duality group $\mathrm{SL}(2,\mathbb{Z})$ of the full type IIB string theory [12]. The next to leading order α'-corrections to the effective action are octic in derivatives of the metric, i.e., fourth order in powers of the Riemann tensor, and receives perturbative contributions only from tree-level and one-loop in the string genus expansion. However, this gives a scalar coefficient in front of the $\mathcal{R}^4$-terms in the effective action which is not $\mathrm{SL}(2,\mathbb{Z})$-invariant. This problem is resolved by noting that there are additional non-perturbative

[1] Strictly speaking, the name U-duality is reserved for the chain of exceptional discrete groups $\mathcal{E}_{n(n)}(\mathbb{Z})$, related to the toroidal compactification of M-theory (see [3] for a review). However, for convenience, we shall in this paper adopt a slight abuse of terminology and refer to any enhanced symmetry group $\mathcal{U}_d(\mathbb{Z})$ as a "U-duality" group. This then applies, for example, to the mapping class group $\mathrm{SL}(n+1,\mathbb{Z})$ of the internal torus in the reduction of pure gravity to three dimensions, and to the T-duality group $\mathrm{SO}(n,n,\mathbb{Z})$ appearing in the reduction of the coupled gravity-2-form system. Moreover, we shall refer to the continuous versions of these groups, $\mathcal{U}_d = \mathcal{U}_d(\mathbb{R})$, as "classical U-duality groups".

[2] One exception being ref. [10] in which the authors considered quadratic curvature corrections to pure gravity in four dimensions. In that special case, the most general correction can be related, through suitable field redefinitions, to the Gauss-Bonnet term which is topological in four dimensions and does not contribute to the dynamics. Hence, the $\mathrm{SL}(2,\mathbb{R})$-symmetry of the compactified Lagrangian is trivially preserved.

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contributions to the octic derivative terms arising from D-instantons ($D(-1)$-branes) [12]. This contribution can be seen as a "completion" of the coefficient to an $\mathrm{SL}(2,\mathbb{Z})$-invariant scalar function which is identified with a certain automorphic function, known as a non-holomorphic Eisenstein series. A weak-coupling (large volume) expansion of this function reproduces the perturbative tree-level and one-loop coefficients at lowest order.

In the scenario described above the completion to a U-duality invariant expression was achieved through the use of a scalar automorphic form, i.e., an automorphic function, which is completely $\mathrm{SL}(2,\mathbb{Z})$-invariant. More generally, one might find terms in the effective action whose non-perturbative completion requires automorphic forms transforming under the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3)$. For example, this was found to be the case in [13], where interaction terms of sixteen fermions were analyzed. These terms transform under the maximal compact subgroup $\mathrm{U}(1) \subset \mathrm{SL}(2,\mathbb{R})$ and so the U-duality invariant completion requires in this case an automorphic form which transform with a $\mathrm{U}(1)$ weight that compensates for the transformation of the fermionic term, and thus renders the effective action invariant.

The need for automorphic forms which transform under the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3)$ was also emphasized in [8], based on the observation that the dilaton exponents in compactified higher curvature corrections correspond to weights of the global symmetry group $\mathcal{U}_3$, implying that these terms transform non-trivially in some representation of $\mathcal{K}(\mathcal{U}_3)$. An explicit realisation of these arguments was found in [11] for the case of compactification on S^1 of the four-dimensional coupled Einstein-Liouville system, supplemented by a four-derivative curvature correction. The resulting effective action was shown to explicitly break the Ehlers $\mathrm{SL}(2,\mathbb{R})$-symmetry; however, an $\mathrm{SL}(2,\mathbb{Z})_{\mathrm{global}} \times \mathrm{U}(1)_{\mathrm{local}}$-invariant effective action was obtained by "lifting" the scalar coefficients to automorphic forms transforming with compensating $\mathrm{U}(1)$ weights. The non-perturbative completion implied by this lifting is in this case attributed to gravitational Taub-NUT instantons [11].

Similar conclusions were drawn in [9], in which compactifications of derivative corrections of second, third and fourth powers of the Riemann tensor were analyzed. Again, it was concluded that the $\mathcal{U}_3$-symmetry is explicitly broken by the correction terms. It was argued, in accordance with the type IIB analysis discussed above, that the result of the compactification – being inherently perturbative in nature — should be considered as the large volume expansion of a $\mathcal{U}_3(\mathbb{Z})$-invariant effective action. It was shown on general grounds that any term resulting from such a compactification can always be lifted to a U-duality invariant expression through the use of automorphic forms transforming in some representation of $\mathcal{K}(\mathcal{U}_3)$.

In this paper we extend some aspects of the analysis of [9]. In [9] only parts of the compactification of the Riemann tensor squared, $\hat{R}_{ABCD}\hat{R}^{ABCD}$, were presented. The terms which were analyzed were sufficient to show that the continuous symmetry was broken, and to argue for the necessity of introducing transforming automorphic forms to restore the U-duality symmetry $\mathcal{U}_3(\mathbb{Z})$. Moreover, the overall volume factor of the internal torus was neglected in the analysis.

We restrict our study to corrections quadratic in the Riemann tensor in order for a complete compactification to be a feasible task. More precisely, we shall focus on a four-derivative correction to the Einstein-Hilbert action in the form of the Gauss-Bonnet term $\hat{R}_{ABCD}\hat{R}^{ABCD} - 4\hat{R}_{AB}\hat{R}^{AB} + \hat{R}^2$. Modulo field equations, this is the only independent invariant quadratic in the Riemann tensor. We extend the investigations of [9] by giving the complete compactification on T^n of the Gauss-Bonnet term from D dimensions to $D-n$ dimensions. In the special case of compactifications to $D-n=3$ dimensions the resulting expression simplifies, making it amenable for a more careful analysis. In particular, one of the main points of this paper is to study the full structure of the dilaton exponents, with the purpose of determining the $\mathrm{sl}(n+1,\mathbb{R})$-representation structure associated with quadratic curvature corrections. In contrast to the general arguments of [7] we have here access to a complete expression after compactification, thus allowing us to perform an exhaustive analysis of the weight structure associated with all terms in the Lagrangian.

We note that effects of adding Gauss-Bonnet correction terms have recently been discussed in the contexts of black hole entropy (see [14] for a recent review and further references) and brane world scenarios (see, e.g., [15]).

1.2 A puzzle and a possible resolution

The research programme outlined above was initially inspired by recent results regarding the question of how curvature corrections in string and M-theory, analyzed close to a spacelike singularity (the "BKL-limit"), fit into the representation structure of the hyperbolic Kac-Moody algebra $E_{10(10)} = \text{Lie } \mathcal{E}_{10(10)}$ [16, 17]. These authors found that generically such curvature corrections are associated with exponents which reside on the negative side of the root lattice of the algebra, indicating that correction terms fall into infinite-dimensional (non-integrable) lowest-weight representations of $E_{10(10)}$.[3] Moreover, it was shown that curvature corrections to eleven-dimensional supergravity match with the root lattice of $E_{10(10)}$ only for the special powers $3k+1,\ k=1,2,3,\dots,$ of the Riemann tensor. This is in perfect agreement with explicit loop calculations, which reveal that the only correction terms with non-zero coefficients are $\mathcal{R}^4, \mathcal{R}^7, \dots,$ etc. [18]. However, when reducing to ten-dimensions and repeating the analysis for type IIA and type IIB supergravity, the restriction on the curvature terms — obtained by requiring compatibility with the $E_{10(10)}$-root lattice — no longer match with known results from string calculations [17]. For example, the $E_{10(10)}$ analysis for type IIA predicts a correction term of order $\mathcal{R}^3$, which is known to be forbidden by supersymmetry. This implies that — even though correct for eleven-dimensional supergravity — the compatibility between higher derivative corrections and the root lattice of $E_{10(10)}$ is clearly not well-understood, and requires refinement.

These results are puzzling also in other respects, most notably because the weights that arise from curvature corrections are *negative* weights of $E_{10(10)}$; with the leading order term in a BKL-like expansion of the $\mathcal{R}^4$-terms being the lowest weight of the representation, and, in fact, corresponds to the negative of a dominant integral weight. This implies that the representation builds upwards and outwards from the interior of the negative fundamental Weyl chamber, rendering the representation non-integrable. From the point of view of

[3]The root lattice of $E_{10(10)}$ is self-dual, implying that the root lattice and the weight lattice coincide. The same is true for $E_{8(8)}$.

the nonlinear sigma model for $\mathcal{E}_{10(10)}/\mathcal{K}(\mathcal{E}_{10(10)})$ this result is also strange, because the correspondence with the tree-level Lagrangian in the BKL-limit requires the use of the Borel gauge, for which no negative weights appear in the Lagrangian [19] (see [20, 21] for reviews). The reason for these puzzling results is essentially due to the "lapse-function" N, representing the reparametrisation invariance in the timelike direction. At tree-level the powers of the lapse-function arising from the measure and from the Ricci scalar cancel, and the remaining exponents correspond to positive roots of $E_{10(10)}$. On the other hand, for terms of higher order in the Riemann tensor there are also higher powers of the lapse-function which "pushes" the exponents to the negative side of the root system.

From a different point of view, similar features have appeared in the analysis of [7]. These authors investigated the general structure of the dilaton exponents upon compactifications on T^8 of quartic curvature corrections to eleven-dimensional supergravity, emphasizing the importance of including the overall "volume factor", which parametrises the volume of the internal torus. Of course, in this case it is the Lie algebra $E_{8(8)} = \text{Lie } \mathcal{E}_{8(8)}$ which is the relevant one, rather than $E_{10(10)}$. However, the inclusion of the volume factor into the dilaton exponents when investigating the weight structure has precisely the same effect as the lapse-function had in the $E_{10(10)}$-case above, namely to push the exponents from the positive root lattice of $E_{8(8)}$ down to the negative root lattice, thus giving rise to negative weights of $E_{8(8)}$.

These results imply that one might use the simpler approach of compactification of curvature corrections to three dimensions in order to develop some intuition regarding the more difficult case of implementing the full $\mathcal{E}_{10(10)}$-symmetry in M-theory. Based on these considerations — and the results obtained in the present paper concerning the representation structure of the compactified Gauss-Bonnet term — we shall in fact argue that the overall volume factor should *not* be included in the analysis of the representation structure. This interpretation draws from the idea that the result of the compactification should be seen as the lowest order term in a large volume expansion of a manifestly U-duality invariant action. From this point of view the volume factor is then associated to the first term in an expansion of an automorphic form of $\mathcal{U}_3(\mathbb{Z})$, transforming in some representation of the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3)$. Moreover, with this interpretation, the dilaton exponents of the compactified quadratic corrections exhibit a more natural structure in terms of representations of $\mathcal{U}_3$. It is our hope that these results can also be applied to the question of how higher derivative corrections to eleven-dimensional supergravity fit into $E_{10(10)}$.

1.3 Organisation of the paper

Our paper is organized as follows. In section 2 we present the result of the compactification of the Gauss-Bonnet term on T^n from D dimensions to $D - n = 3$ dimensions. The completely general action representing the compactification to arbitrary dimensions is given in appendix A. The result in three dimensions is given in section 2 after dualisation of all Kaluza-Klein vectors into scalars, which is the case of most interest from the U-duality point of view. We then proceed in section 3 with the analysis of the compactified Lagrangian. We analyze in detail the dilaton exponents in terms of the representation theory of $\mathrm{sl}(n+1, \mathbb{R})$, which is the enhanced symmetry group of the compactified tree-level Lagrangian. Finally, in section 4 we suggest a possible non-perturbative completion of the compactified Lagrangian into a manifestly U-duality invariant expression. We explain how this completion requires the lifting of the coefficients in the Lagrangian into automorphic forms transforming non-trivially under the maximal compact subgroup $\mathcal{K}(\mathcal{U}_3) \subset \mathcal{U}_3$. We interpret our results and provide a comparison with the existing literature. All calculational details are displayed in appendix A.

2. Compactification of the Gauss-Bonnet term

In this section we outline the derivation of the toroidal compactification of the Gauss-Bonnet term from D dimensions to $D-n$ dimensions. In eq. (A.22) of appendix A we give the full result for the compactification to arbitrary dimensions. Here we focus on the special case of $D-n=3$, which is the most relevant case for the questions we pursue in this paper.

2.1 The general procedure

The Gauss-Bonnet Lagrangian density is quadratic in the Riemann tensor and takes the explicit form

$$\mathcal{L}_{\text{GB}} = \hat{e}\big[\hat{R}_{ABCD}\hat{R}^{ABCD} - 4\hat{R}_{AB}\hat{R}^{AB} + \hat{R}^2\big]. \tag{2.1}$$

The compactification of the D-dimensional Riemann tensor $\hat{R}^A_{BCD}$ on an n-torus, T^n, is done in three steps: first we perform a Weyl-rescaling of the total vielbein, followed by a splitting of the external and internal indices, and finally we define the parametrisation of the internal vielbein. In the following we shall always assume that the torsion vanishes.

Conventions and reduction Ansatz. Our index conventions are as follows. $M, N, \dots$ denote D dimensional curved indices, and $A, B, \dots$ denote D dimensional flat indices. Upon compactification we split the indices according to $M = (\mu, m)$, where $\mu, \nu, \dots$ and $m, n, \dots$ are curved external and internal indices, respectively. Similarly, the flat indices split into external and internal parts according to $A = (\alpha, a)$.

Our reduction Ansatz for the vielbein is

$$\hat{e}_M{}^A = e^{\varphi}\tilde{e}_M{}^A = e^{\varphi}\begin{pmatrix} e_\mu{}^\alpha & \mathcal{A}_\mu^m \tilde{e}_m{}^a \\ 0 & \tilde{e}_m{}^a \end{pmatrix}, \tag{2.2}$$

where the internal vielbein $\tilde{e}_m{}^a$ is an element of the isometry group $\mathrm{GL}(n,\mathbb{R})$ of the n-torus. Later on we shall parametrise $\tilde{e}_m{}^a$ in various ways. With this Ansatz, the line element becomes

$$ds_D^2 = e^{2\varphi}\Big\{ds_{D-n}^2 + \big[(dx^m + \mathcal{A}^m_{(1)})\tilde{e}_m{}^a\big]^2\Big\}. \tag{2.3}$$

Weyl-Rescaling. In order to obtain a Lagrangian in Einstein frame after dimensional reduction, we perform a Weyl-rescaling of the D-dimensional vielbein,

$$\hat{e}_M{}^A \longrightarrow \tilde{e}_M{}^A = e^{-\varphi}\hat{e}_M{}^A. \tag{2.4}$$

Note that all D-dimensional objects before rescaling are denoted $\hat{X}$, the Weyl-rescaled objects are denoted $\tilde{X}$, while the $d = (D-n)$-dimensional objects are written without any diacritics. After the Weyl-rescaling the Gauss-Bonnet Lagrangian, including the volume measure $\hat{e} = e^{D\varphi}\tilde{e}$, can be conveniently organized in terms of equations of motion and total derivatives. This is achieved using integration by parts, where $\tilde{\nabla}_{(A}\tilde{\partial}_{B)}\varphi$ does not appear explicitly. The resulting Lagrangian is (see appendix A):

$$\begin{aligned}
\mathcal{L}_{\rm GB} = \tilde{e}e^{(D-4)\varphi}\Bigg\{&\tilde{R}^2_{\rm GB} - (D-3)(D-4)\Big[2(D-2)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi + (D-2)(D-3)(\tilde{\partial}\varphi)^4\\
&+4\Big(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\Big)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi)\Big]\Bigg\}\\
+2(D-3)\tilde{e}\tilde{\nabla}_A\Bigg\{&e^{(D-4)\varphi}\Big[(D-2)^2(\tilde{\partial}\varphi)^2\tilde{\partial}^A\varphi + 2(D-2)(\tilde{\Box}\varphi)\tilde{\partial}^A\varphi\\
&-(D-2)\tilde{\partial}^A(\tilde{\partial}\varphi)^2 + 4\Big(\tilde{R}^{AB} - \frac{1}{2}\eta^{AB}\tilde{R}\Big)\tilde{\partial}_B\varphi\Big]\Bigg\},
\end{aligned} \tag{2.5}$$

where $\tilde{R}^2_{GB}$ represents the rescaled Gauss-Bonnet combination. In $D = 4$ the Lagrangian is only altered by a total derivative, while in $D = 3$ the Lagrangian it is merely rescaled by a factor of $e^{-\varphi}$. The total derivative terms here will remain total derivatives even after the compactification. Along with the volume factor the Weyl-rescaling will determine the overall exponential dilaton factor, which shall play an important role in the analysis that follows.

2.2 Tree-level scalar coset symmetries

The internal vielbein $\hat{e}_m{}^a$ can be used to construct the internal metric $\hat{g}_{mn} = \hat{e}_m{}^a\hat{e}_n{}^b\delta_{ab}$, which is manifestly invariant under local SO(n) rotations in the reduced directions. Thus we are free to fix a gauge for the internal vielbein using the SO(n)-invariance. After compactification the volume measure becomes $\tilde{e} = e\tilde{e}_{\rm int}$, where e is the determinant of the spacetime vielbein and $\tilde{e}_{\rm int}$ is the determinant of the internal vielbein. Defining the Weyl-rescaling coefficient as $e^{-(D-2)\varphi} \equiv \tilde{e}_{\rm int}$ ensures that the reduced Lagrangian is in Einstein frame.

The GL($n, \mathbb{R}$) group element $\tilde{e}_m{}^a$ can now be parameterized in several ways, and we will discuss the two most natural choices here. The first choice is included for completeness, while it is the second choice which we shall subsequently employ in the compactification of the Gauss-Bonnet term.

First parametrisation — Making the symmetry manifest. First, there is the possibility of separating out the determinant of the internal vielbein according to $\tilde{e}_m{}^a = (\tilde{e}_{\rm int})^{1/n}\varepsilon_m{}^a = e^{-\frac{(D-2)}{n}\varphi}\varepsilon_m{}^a$, where $\varepsilon_m{}^a$ is an element of SL($n, \mathbb{R}$) in any preferred gauge. The line element takes the form

$$ds^2_D = e^{2\varphi}\left\{ds^2_{D-n} + e^{-2\frac{(D-2)}{n}\varphi}\left[(dx^m + \mathcal{A}^m_{(1)})\varepsilon_m{}^a\right]^2\right\}. \tag{2.6}$$

This Ansatz is nice for investigating the symmetry properties of the reduced Lagrangian because the GL($n, \mathbb{R}$)-symmetry of the internal torus is manifestly built into the formalism.

More precisely, the reduction of the tree-level Einstein-Hilbert Lagrangian, $\hat{e}\hat{R}$, to $d=D-n$ dimensions becomes,

$$\mathcal{L}_{\text{EH}}^{[d]} = e\left[R - \frac{1}{4}e^{-2\frac{(D-2)}{n}\xi\rho}F_{c\alpha\beta}F^{c\alpha\beta} - \frac{1}{2}(\partial\rho)^2 - \text{tr}(P_\alpha P^\alpha) - 2\xi\Box\rho\right], \tag{2.7}$$

where $F^c{}_{\alpha\beta} \equiv \varepsilon_m{}^a F^m{}_{\alpha\beta}$ and

$$P_\alpha{}^{bc} \equiv \varepsilon^{m(b}\partial_\alpha \varepsilon_m{}^{c)} = \tilde{P}_\alpha{}^{bc} + \frac{(D-2)}{n}\xi\partial_\alpha\rho\delta^{bc}. \tag{2.8}$$

Notice that $P_\alpha{}^{bc}$ is $\text{sl}(n,\mathbb{R})$ valued and hence fulfills $\text{tr}(P_\alpha)=0$. To obtain eq. (2.7) we also performed a scaling $\varphi = \xi\rho$ with $\xi = \sqrt{\frac{n}{2(D-2)(D-n-2)}}$, so as to ensure that the scalar field ρ appears canonically normalized in the Lagrangian.

The $\text{SL}(n,\mathbb{R})$-symmetry is manifest in this Lagrangian because the term $\text{tr}(P_\alpha P^\alpha)$ is constructed using the invariant Killing form on $\text{sl}(n,\mathbb{R})$. By dualising the two-form field strength $F_{(2)}$, the symmetry is enhanced to $\text{SL}(n+1,\mathbb{R})$. With a slight abuse of terminology we call this the (classical) "U-duality" group. Since we are only investigating the pure gravity sector, this is of course only a subgroup of the full continuous U-duality group.

It was already shown in [9], that the tree-level symmetry $\text{SL}(n+1,\mathbb{R})$ is not realized in the compactified Gauss-Bonnet Lagrangian. It was argued, however, that the quantum symmetry $\text{SL}(n+1,\mathbb{Z})$ could be reinstated by "lifting" the result of the compactification through the use of automorphic forms. In this paper we take the same point of view, but since we now have access to the complete expression of the compactified Gauss-Bonnet Lagrangian we can here extend the analysis of [9] in some aspects. In order to do this we shall make use of a different parametrisation than the one displayed above, which illuminates the structure of the dilaton exponents in the Lagrangian. The dilaton exponents reveals the weight structure of the global symmetry group and so can give information regarding which representation of the U-duality group we are dealing with.

Second parametrisation — Revealing the root structure. The second natural choice of the internal vielbein is to parameterize it in triangular form by using dimension by dimension compactification [22–24]. Instead of extracting only the determinant of the vielbein, one dilaton scalar is pulled out for each compactified dimension according to $\tilde{e}_m{}^a = e^{-\frac{1}{2}\vec{f}_a\cdot\vec{\phi}}u_m{}^a$, where $\vec{\phi} = (\phi_1,\ldots,\phi_n)$ and

$$\vec{f}_a = 2(\alpha_1,\ldots,\alpha_{a-1},(D-n-2+a)\alpha_a,\underbrace{0,\ldots,0}_{n-a}), \tag{2.9}$$

with

$$\alpha_a = \frac{1}{\sqrt{2(D-n-2+a)(D-n-3+a)}}. \tag{2.10}$$

The internal vielbein is now the Borel representative of the coset $\text{GL}(n,\mathbb{R})/\text{SO}(n)$, with the diagonal degrees of freedom $e^{-\frac{1}{2}\vec{f}_a\cdot\vec{\phi}}$ corresponding to the Cartan generators and the upper triangular degrees of freedom

$$u_m{}^a = [(1-\mathcal{A}_{(0)})^{-1}]_m{}^a = [1+\mathcal{A}_{(0)}+(\mathcal{A}_{(0)})^2+\ldots]_m{}^a \tag{2.11}$$

corresponding to the positive root generators. The form of eq. (2.11) follows naturally from a step by step compactification, where the scalar potentials $(\mathcal{A}_{(0)})^i_j$, arising from the compactification of the graviphotons, are nonzero only when $i > j$. The sum of the vectors $\vec{f}_a$ can be shown to be

$$\sum_{a=1}^{n} \vec{f}_a = \frac{D-2}{3}\vec{g}, \tag{2.12}$$

$\vec{g} \equiv 6(\alpha_1, \alpha_2 \ldots, \alpha_n)$. In addition, $\vec{g}$ and $\vec{f}_a$ obey

$$\begin{aligned} \vec{g}\cdot\vec{g} &= \frac{18n}{(D-2)(D-n-2)}, \\ \vec{g}\cdot\vec{f}_a &= \frac{6}{D-n-2}, \\ \vec{f}_a\cdot\vec{f}_b &= 2\delta_{ab} + \frac{2}{D-n-2}, \end{aligned} \tag{2.13}$$

and

$$\sum_{a=1}^{n}(\vec{f}_a\cdot\vec{x})(\vec{f}_a\cdot\vec{y}) = 2(\vec{x}\cdot\vec{y}) + \frac{D-2}{9}(\vec{g}\cdot\vec{x})(\vec{g}\cdot\vec{y}). \tag{2.14}$$

These scalar products can naturally be used to define the Cartan matrix, once a set of simple root vectors are found. The line element becomes

$$ds^2_D = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}}\Big\{ds^2_{D-n} + \sum_{a=1}^{n} e^{-\vec{f}_a\cdot\vec{\phi}}\left[(dx^m + \mathcal{A}^m_{(1)})u_m{}^a\right]^2\Big\}, \tag{2.15}$$

yielding the corresponding Einstein-Hilbert Lagrangian in d dimensions

$$\mathcal{L}^{[d]}_{\mathrm{EH}} = e\left[R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}\sum_{a=1}^{n} e^{-\vec{f}_a\cdot\vec{\phi}}F_{a\beta\gamma}F^{a\beta\gamma} - \frac{1}{2}\sum_{\substack{b,c=1\\ b<c}}^{n} e^{(\vec{f}_b-\vec{f}_c)\cdot\vec{\phi}}G_{\alpha bc}G^{\alpha bc} - \frac{1}{3}\vec{g}\cdot\Box\vec{\phi}\right], \tag{2.16}$$

with $F^c{}_{\alpha\beta} \equiv u_m{}^a F^m{}_{\alpha\beta}$ and

$$G_\alpha{}^{bc} = u^{mb}\partial_\alpha u_m{}^c = e^{-\frac{1}{2}(\vec{f}_b-\vec{f}_c)\cdot\vec{\phi}}\left[\left(\tilde{P}_\alpha{}^{bc} + \frac{1}{2}\vec{f}_b\cdot\partial_\alpha\vec{\phi}\delta^{bc}\right) + Q_\alpha{}^{bc}\right]. \tag{2.17}$$

Here, no Einstein's summation rule is assumed for the flat internal indices. Notice also that $G_\alpha{}^{bc}$ is non-zero only when $b < c$.

We shall refer to the various exponents of the form $e^{\vec{x}\cdot\vec{\phi}}$ ($\vec{x}$ being some vector in $\mathbb{R}^n$) collectively as "dilaton exponents". If relevant, this also includes the contribution from the overall volume factor.

All the results obtained in this parametrisation can be converted to the first parametrisation simply by using the following identifications

$$\begin{aligned} \frac{1}{3}(\vec{g}\cdot\vec{\phi}) &= 2\xi\rho, \\ \vec{f}_a\cdot\vec{\phi} &= 2\frac{(D-2)}{n}\xi\rho, \qquad \forall a, \\ \vec{\phi}\cdot\vec{\phi} &= \rho^2, \end{aligned} \tag{2.18}$$

where one should keep in mind that $\xi = \sqrt{\frac{n}{2(D-2)(D-n-2)}}$. Notice also that our compactification procedure breaks down at $D-n=2$, in which case the scalar products in eq. (2.13) become ill-defined.

Even though proving the symmetry contained in the Lagrangian is somewhat more cumbersome compared to the first choice of parametrisation, since all the group actions have to be carried out adjointly in a formal manner, the second choice comes to its power when dealing with the exceptional symmetry groups of the supergravities for which no matrix representations exist. This parametrisation is particularly suitable for reading off the root vectors of the underlying symmetry algebra; they appear as exponential factors in front of each term in the Lagrangian. Identifying a complete set of root vectors in this way gives a necessary but not sufficient constraint on the underlying symmetry.

2.3 The Gauss-Bonnet lagrangian reduced to three dimensions

When reducing to $D-n=3$ dimensions, we can dualise the two-form field strength $\tilde{F}^a{}_{\alpha\beta} \equiv \tilde{e}_m{}^a F^m{}_{\alpha\beta}$ of the graviphoton $\mathcal{A}_{(1)}$ into the one-form $\tilde{H}_{a\alpha}$. More explicitly, we employ the standard dualisation

$$\delta_{ab}\tilde{F}^b{}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\tilde{e}^m{}_a\partial^\gamma\chi_m \equiv \epsilon_{\alpha\beta\gamma}\tilde{H}_a{}^\gamma. \tag{2.19}$$

When we go to Einstein frame, the appearance of the inverse vielbein $\tilde{e}^m{}_a$ in the definition of the one-form $\tilde{H}_{a\alpha}$ implies there is a sign flip on its dilaton exponent in the Lagrangian after dualisation. The dualisation presented here follows from the tree-level Lagrangian, but in general receives higher order α'-corrections. However, these lead to terms of higher derivative order than quartic and so can be neglected in the present analysis [7, 9].

The compactification is performed according to the standard procedure by separating the indices; the detailed calculations can be found in appendix A. The final results are written in such way that the only explicit derivative terms appearing are divergences, total derivatives and first derivatives on the dilatons φ. The complete compactification of the Gauss-Bonnet Lagrangian on T^n to arbitrary dimensions $D-n$ is given in eq. (A.22) of appendix A.[4] This expression is rather messy and difficult to work with. However, by making use of all first order equations of motion, dualising all graviphotons to scalars, and restricting to $D-n=3$, the Lagrangian simplifies considerably. The end result reads

$$\begin{aligned}\mathcal{L}^{[3]}_{\mathrm{GB}} = \sqrt{|g|}e^{-2\varphi}\Big\{ &-\frac{1}{4}\tilde{H}_{a\gamma}\tilde{H}_b{}^\gamma\tilde{H}^a{}_\delta\tilde{H}^{b\delta} + \frac{1}{4}\tilde{H}^2\tilde{H}^2 - 4\tilde{H}^2(\partial\varphi)^2 + 2\tilde{H}^{c\alpha}\tilde{P}_{\alpha cd}\tilde{P}^{\beta de}\tilde{H}_{e\beta} \\ &-2\tilde{H}^{c\alpha}\tilde{P}_{\beta cd}\tilde{P}^{\beta de}\tilde{H}_{e\alpha} + 4\tilde{H}_{c\alpha}\tilde{P}^{\alpha cd}\tilde{H}_d{}^\beta\partial_\beta\varphi - 6\tilde{H}_{c\alpha}\tilde{P}^{\beta cd}\tilde{H}_d{}^\alpha\partial_\beta\varphi \\ &+2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta) + 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - (\tilde{P}^2)^2 + 8\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\beta)\partial^\alpha\varphi \\ &-4(D-2)\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\partial^\alpha\varphi\partial^\beta\varphi + 2(D+2)\tilde{P}^2(\partial\varphi)^2 + (D-2)(D-4)(\partial\varphi)^2(\partial\varphi)^2\Big\},\end{aligned} \tag{2.20}$$

[4] Kaluza-Klein reduction of quadratic curvature corrections has also been analyzed from a different point of view in [25].

where $\tilde{H}^2 \equiv \tilde{H}_{\alpha\beta}\tilde{H}^{\alpha\beta}$ and $\tilde{P}^2 \equiv \tilde{P}_{\alpha bc}\tilde{P}^{\alpha bc}$. Note that contributions from the boundary terms and terms proportional to the equations of motion have been ignored. The one-form $\tilde{P}_\alpha$ is the Maurer-Cartan form associated with the internal vielbein $\tilde{e}_m^{\ a}$, and so takes values in the Lie algebra $\mathrm{gl}(n,\mathbb{R}) = \mathrm{sl}(n,\mathbb{R}) \oplus \mathbb{R}$. Here, the abelian summand $\mathbb{R}$ corresponds to the "trace-part" of $\tilde{P}_\alpha$. Explicitly, we have $\mathrm{tr}(\tilde{P}_\alpha) = -(D-2)\partial_\alpha\varphi$. We shall discuss various properties of $\tilde{P}_\alpha$ in more detail below.

Finally, we note that the three-dimensional Gauss-Bonnet term is absent from the reduced Lagrangian because it vanishes identically in three dimensions:

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 = 0, \qquad (\alpha,\beta,\gamma,\delta = 1,2,3). \tag{2.21}$$

The remainder of this paper is devoted to a detailed analysis of the symmetry properties of eq. (2.20).

3. Algebraic structure of the compactified Gauss-Bonnet term

We have seen that the Ansatz presented in eq. (2.15) is particularly suitable for identifying the roots of the relevant symmetry algebra from the dilaton exponents associated with the diagonal components of the internal vielbein. Through this analysis one may deduce that for the lowest order effective action, the terms in the action are organized according to the adjoint representation of $\mathrm{sl}(n+1,\mathbb{R})$, for which the weights are the roots. The aim of this section is to extend the analysis to the Gauss-Bonnet Lagrangian. By general arguments [7, 8], it has been shown that the exponents no longer correspond to roots of the symmetry algebra but rather they now lie on the weight lattice. Here, however, we have access to the complete compactified Lagrangian and we may therefore present an explicit counting of the weights in the dilaton exponents and identify the relevant $\mathrm{sl}(n+1,\mathbb{R})$-representation.

An exhaustive analysis of the $\mathrm{sl}(4,\mathbb{R})$-representation structure of the Gauss-Bonnet term compactified from 6 to 3 dimensions on T^3 is performed. We do this in two alternative ways.

First, we neglect the contribution from the overall dilaton factor $e^{-2\varphi}$ in the representation structure. This is consistent before dualisation because this factor is $\mathrm{SL}(3,\mathbb{R})$-invariant. However, after dualisation this is no longer true and one must understand what role this factor plays in the algebraic structure. If one continues to neglect this factor then all the weights fit into the **84**-representation of $\mathrm{sl}(4,\mathbb{R})$ with Dynkin labels $[2,0,2]$.

On the other hand, including the overall exponential dilaton factor in the weight structure induces a shift on the weights so that the highest weight is associated with the **36**-representation of $\mathrm{sl}(4,\mathbb{R})$ instead, with Dynkin labels $[2,0,1]$. However, this representation is not "big enough" to incorporate all the weights in the Lagrangian. It turns out that there are additional weights outside of the **36** that fit into a **27**-representation of $\mathrm{sl}(3,\mathbb{R})$. Unfortunately there seems to be no obvious argument for which $\mathrm{sl}(4,\mathbb{R})$-representation those "extra" weights should belong to.

This indicates that the first approach, where the dilaton pre-factor is neglected, is the correct way to interpret the result of the compactification because then all weights are "unified" in a single representation of the U-duality group. A detailed demonstration of this follows below.

3.1 Kaluza-Klein reduction and $\mathrm{sl}(n,\mathbb{R})$-representations

We shall begin by rewriting the reduction Ansatz in a way which has a more firm Lie algebraic interpretation. Recall from eq. (2.15) that the standard Kaluza-Klein Ansatz for the metric is

$$ds_D^2 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} ds_d^2 + e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} \sum_{i=1}^{n} e^{-\vec{f}_i\cdot\vec{\phi}} \left[\left(dx^m + \mathcal{A}^m_{(1)}\right) u_m{}^a\right]^2. \tag{3.1}$$

The exponents in this Ansatz are linear forms on the space of dilatons. Let $\vec{e}_i,\ i = 1,\ldots,n$, constitute an n-dimensional orthogonal basis of $\mathbb{R}^n$,

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}. \tag{3.2}$$

Since there is a non-degenerate metric on the space of dilatons (the Cartan subalgebra $\mathfrak{h} \subset \mathrm{sl}(n+1,\mathbb{R})$) we can use this to identify this space with its dual space of linear forms. Thus, we may express all exponents in the orthogonal basis $\vec{e}_i$ and the vectors $\vec{f}_i$ and $\vec{g}$ may then be written as

$$\begin{aligned} \vec{f}_i &= \sqrt{2}\vec{e}_i + \alpha\vec{g}, \\ \vec{g} &= \beta\sum_{i=1}^{n}\vec{e}_i, \end{aligned} \tag{3.3}$$

where the constants α and β are defined as

$$\begin{aligned} \alpha &= \frac{1}{3n}\left(D-2-\sqrt{(D-n-2)(D-2)}\right), \\ \beta &= \sqrt{\frac{18}{(D-n-2)(D-2)}}. \end{aligned} \tag{3.4}$$

Note here that the constant α is not the same as the α_a of eq. (2.9).

The combinations

$$\vec{f}_i - \vec{f}_j = \sqrt{2}\vec{e}_i - \sqrt{2}\vec{e}_j \tag{3.5}$$

span an $(n-1)$-dimensional lattice which can be identified with the root lattice of $A_{n-1} = \mathrm{sl}(n,\mathbb{R})$. For compactification of the pure Einstein-Hilbert action to three dimensions, the dilaton exponents precisely organize into the complete set of positive roots of $\mathrm{sl}(n,\mathbb{R})$, revealing that it is the adjoint representation which is the relevant one for the U-duality symmetries of the lowest order (two-derivative) action. After dualisation of the Kaluza-Klein one forms $\mathcal{A}_{(1)}$ the symmetry is lifted to the full adjoint representation of $\mathrm{sl}(n+1,\mathbb{R})$.

When we compactify higher derivative corrections to the Einstein-Hilbert action it is natural to expect that other representations of $\mathrm{sl}(n,\mathbb{R})$ and $\mathrm{sl}(n+1,\mathbb{R})$ become relevant. In order to pursue this question for the Gauss-Bonnet Lagrangian, we shall need some features of the representation theory of $\mathrm{sl}(n+1,\mathbb{R})$.

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Representation theory of $A_n = \mathrm{sl}(n+1,\mathbb{R})$. For the infinite class of simple Lie algebras A_n, it is possible to choose an embedding of the weight space $\mathfrak{h}^\star$ in $\mathbb{R}^{n+1}$ such that $\mathfrak{h}^\star$ is isomorphic to the subspace of $\mathbb{R}^{n+1}$ which is orthogonal to the vector $\sum_{i=1}^{n+1} \vec{e}_i$ (see, e.g., [26]). We can use this fact to construct an embedding of the $(n-1)$-dimensional weight space of $A_{n-1} = \mathrm{sl}(n,\mathbb{R})$ into the n-dimensional weight space of $A_n = \mathrm{sl}(n+1,\mathbb{R})$, in terms of the n basis vectors $\vec{e}_i$ of $\mathbb{R}^n$.

To this end we define the new vectors

$$\begin{aligned}\vec{\omega}_i &= \vec{f}_i - \left(\alpha + \frac{\sqrt{2}}{n\beta}\right)\vec{g} \\ &= \sqrt{2}\vec{e}_i - \frac{\sqrt{2}}{n}\sum_{j=1}^{n}\vec{e}_j,\end{aligned} \tag{3.6}$$

which have the property that

$$\vec{\omega}_i \cdot \vec{g} = \sqrt{2}\beta - \sqrt{2}\beta = 0. \tag{3.7}$$

This implies that the vectors $\vec{\omega}_i$ form a (non-orthogonal) basis of the $(n-1)$-dimensional subspace $U \subset \mathbb{R}^n$, orthogonal to $\vec{g}$. The space U is then isomorphic to the weight space $\mathfrak{h}^\star$ of $A_{n-1} = \mathrm{sl}(n,\mathbb{R})$. Since there are n vectors $\vec{\omega}_i$, this basis is overcomplete. However, it is easy to see that not all $\vec{\omega}_i$ are independent, but are subject to the relation

$$\sum_{i=1}^{n} \vec{\omega}_i = 0. \tag{3.8}$$

A basis of simple roots of $\mathfrak{h}^\star$ can now be written in three alternative ways

$$\vec{\alpha}_i = \vec{f}_i - \vec{f}_{i+1} = \vec{\omega}_i - \vec{\omega}_{i+1} = \sqrt{2}(\vec{e}_i - \vec{e}_{i+1}), \qquad (i = 1,\ldots,n-1). \tag{3.9}$$

What is the algebraic interpretation of the vectors $\vec{\omega}_i$? It turns out that they may be identified with the weights of the n-dimensional fundamental representation of $\mathrm{sl}(n,\mathbb{R})$. The condition $\sum_{i=1}^{n} \vec{\omega}_i = 0$ then reflects the fact that the generators of the fundamental representation are traceless.

In addition, we can use the weights of the fundamental representation to construct the fundamental weights $\vec{\Lambda}_i$, defined by

$$\vec{\alpha}_i \cdot \vec{\Lambda}_j = 2\delta_{ij}. \tag{3.10}$$

One finds

$$\vec{\Lambda}_i = \sum_{j=1}^{i} \vec{\omega}_j, \qquad (i = 1,\ldots,n-1), \tag{3.11}$$

which can be seen to satisfy eq. (3.10).

The relation, eq. (3.11), between the fundamental weights $\vec{\Lambda}_i$ and the weights of the fundamental representation $\vec{\omega}_i$ can be inverted to

$$\vec{\omega}_i = \vec{\Lambda}_i - \vec{\Lambda}_{i-1}, \qquad (i = 1,\ldots,n-1). \tag{3.12}$$

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In addition, the n:th weight is

$$\vec{\omega}_n = -\vec{\Lambda}_{n-1}, \tag{3.13}$$

corresponding to the lowest weight of the fundamental representation.

We may now rewrite the Kaluza-Klein Ansatz in a way such that the weights $\vec{\omega}_i$ appear explicitly in the metric[5]

$$ds_D^2 = e^{\frac{1}{3}\vec{g}\cdot\vec{\phi}} ds_d^2 + e^{\gamma\vec{g}\cdot\vec{\phi}} \sum_{i=1}^{n} e^{-\vec{\omega}_i\cdot\vec{\phi}} \Big[\big(dx^m + \mathcal{A}^m_{(1)}\big) u_m{}^a\Big]^2, \tag{3.14}$$

with

$$\gamma = \frac{1}{3} - \alpha - \frac{\sqrt{2}}{n\beta}. \tag{3.15}$$

3.2 The algebraic structure of Gauss-Bonnet in three dimensions

We are interested in the dilaton exponents in the scalar part of the three-dimensional Lagrangian. For the Einstein-Hilbert action we know that these are of the forms

$$\vec{f}_a - \vec{f}_b \quad (b > a), \qquad \text{and} \qquad \vec{f}_a. \tag{3.16}$$

The first set of exponents $\vec{f}_a - \vec{f}_b$ correspond to the positive roots of $\mathrm{sl}(n, \mathbb{R})$ and the second set $\vec{f}_a$, which contributes to the scalar sector after dualisation, extends the algebraic structure to include all positive roots of $\mathrm{sl}(n+1, \mathbb{R})$. The highest weight $\vec{\lambda}^{\mathrm{hw}}_{\mathrm{ad},n}$ of the adjoint representation of $A_n = \mathrm{sl}(n+1, \mathbb{R})$ can be expressed in terms of the fundamental weights as

$$\vec{\lambda}^{\mathrm{hw}}_{\mathrm{ad},n} = \vec{\Lambda}_1 + \vec{\Lambda}_n, \tag{3.17}$$

corresponding to the Dynkin labels

$$[1, 0, \ldots, 0, 1].$$

We see that before dualisation the highest weight of the adjoint representation of $\mathrm{sl}(n, \mathbb{R})$ arises in the dilaton exponents in the form $\vec{f}_1 - \vec{f}_n = \vec{\omega}_1 - \vec{\omega}_n = \vec{\Lambda}_1 + \vec{\Lambda}_{n-1} = \vec{\lambda}^{\mathrm{hw}}_{\mathrm{ad},n-1}$.

We proceed now to analyze the various dilaton exponents arising from the Gauss-Bonnet term after compactification to three dimensions. These can be extracted from each term in the Lagrangian eq. (2.20) by factoring out the diagonal components of the internal vielbein according to $\tilde{e}_m{}^a = e^{-\frac{1}{2}\vec{f}_a\cdot\vec{\phi}} u_m{}^a$. For example, before dualisation we have the manifestly $\mathrm{SL}(n, \mathbb{R})$-invariant term $\mathrm{tr}(\tilde{P}_\alpha \tilde{P}_\beta \tilde{P}^\alpha \tilde{P}^\beta)$. Expanding this gives (among others) the following types of terms

$$\begin{aligned} \mathrm{tr}\big(\tilde{P}_\alpha \tilde{P}_\beta \tilde{P}^\alpha \tilde{P}^\beta\big) \sim & \sum_{\substack{b\,<\,a,\,c \\ d\,<\,a,\,c}} e^{-(\vec{f}_a - \vec{f}_b + \vec{f}_c - \vec{f}_d)\cdot\vec{\phi}} G_{\alpha ba} G_\beta{}^{bc} G^\alpha{}_{dc} G^{\beta da} + \cdots \\ & + \sum_{a\,<\,c\,<\,d\,<\,b} e^{(\vec{f}_a - \vec{f}_b)\cdot\vec{\phi}} G_{\alpha ab} G_\beta{}^{ac} G^\alpha{}_{cd} G^{\beta db} + \cdots . \end{aligned} \tag{3.18}$$

[5]A similar construction was given in [27].

After dualisation, we need to take into account also terms containing $\tilde{H}^{\alpha}{}_{a}$. We have then, for example, the term

$$\tilde{H}^4 \sim \sum_{a,b} e^{(\vec{f}_a+\vec{f}_b)\cdot\vec{\phi}} H^4. \tag{3.19}$$

Many different terms in the Lagrangian might in this way give rise to the same dilaton exponents. As can be seen from eq. (3.18), the internal index contractions yield constraints on the various exponents. We list below all the "independent" exponents, i.e., those which are the least constrained. All other exponents follow as special cases of these. Before dualisation we find the following exponents:

$$\begin{aligned} &\vec{f}_a - \vec{f}_b \qquad (b > a), \\ &\vec{f}_c + \vec{f}_d - \vec{f}_a - \vec{f}_b \qquad (c < a,\ c < b,\ d < a,\ d < b), \\ &\vec{f}_a + \vec{f}_b - \vec{f}_c - \vec{f}_d \qquad (b < c,\ a < d), \end{aligned} \tag{3.20}$$

and after dualisation we also get contributions from

$$\begin{aligned} &\vec{f}_a, \\ &\vec{f}_a + \vec{f}_b, \\ &\vec{f}_a + \vec{f}_b - \vec{f}_c, \qquad (a < c,\ b < c). \end{aligned} \tag{3.21}$$

Let us first investigate the general weight structure of the dilaton exponents before dualisation. The highest weight arises from the terms of the form $\vec{f}_c + \vec{f}_d - \vec{f}_a - \vec{f}_b$ when $c = d = 1$ and $a = b = n$, i.e., for the dilaton vector $2\vec{f}_1 - 2\vec{f}_n$. This can be written in terms of the fundamental weights as follows

$$2\vec{f}_1 - 2\vec{f}_n = 2\vec{\omega}_1 - 2\vec{\omega}_n = 2\vec{\Lambda}_1 + 2\vec{\Lambda}_{n-1}, \tag{3.22}$$

which is the highest weight of the $[2, 0, \ldots, 0, 2]$-representation of $\mathrm{sl}(n, \mathbb{R})$.

3.3 Special case: compactification from $D = 6$ on T^3

In order to determine if this is indeed the correct representation for the Gauss-Bonnet term, we shall now restrict to the case of $n = 3$, i.e., compactification from $D = 6$ on T^3. We do this so that a complete counting of the weights in the Lagrangian is a tractable task.[6] Before dualisation we then expect to find the representation **27** of $\mathrm{sl}(3, \mathbb{R})$, with Dynkin labels $[2, 2]$. We will see that, after dualisation, this representation lifts to the representation **84** of $\mathrm{sl}(4, \mathbb{R})$, with Dynkin labels $[2, 0, 2]$.

It is important to realize that of course the Lagrangian will not display the complete set of weights in these representations, but only the *positive* weights, i.e., the ones that can be obtained by summing positive roots only. Let us begin by analyzing the weight structure before dualisation. From eq. (3.20) we find the weights

$$\begin{array}{lll} \vec{f}_1 - \vec{f}_2, & \vec{f}_2 - \vec{f}_3, & \vec{f}_1 - \vec{f}_3, \\ 2(\vec{f}_1 - \vec{f}_2), & 2(\vec{f}_2 - \vec{f}_3), & 2(\vec{f}_1 - \vec{f}_3), \\ 2\vec{f}_1 - \vec{f}_2 - \vec{f}_3, & \vec{f}_1 + \vec{f}_2 - 2\vec{f}_3. & \end{array} \tag{3.23}$$

[6] The simpler case where one performs compactification from $D = 5$ on T^2 behaves in a similar fashion, where we find a level decomposition of $\mathrm{sl}(3, \mathbb{R})$ in terms of $\mathrm{sl}(2, \mathbb{R})$.

The first three may be identified with the positive roots of $\mathrm{sl}(3,\mathbb{R})$, $\vec{\alpha}_1 = \vec{f}_1 - \vec{f}_2$, $\vec{\alpha}_2 = \vec{f}_2 - \vec{f}_3$ and $\vec{\alpha}_\theta = \vec{f}_1 - \vec{f}_3$. The second line then corresponds to $2\vec{\alpha}_1$, $2\vec{\alpha}_2$ and $2\vec{\alpha}_\theta$. The remaining weights are

$$\begin{aligned} \vec{f}_1 + \vec{f}_2 - 2\vec{f}_3 &= \vec{\alpha}_1 + 2\vec{\alpha}_2, \\ 2\vec{f}_1 - \vec{f}_2 - \vec{f}_3 &= 2\vec{\alpha}_1 + \vec{\alpha}_2. \end{aligned} \tag{3.24}$$

These weights are precisely the eight positive weights of the **27** representation of $\mathrm{sl}(3,\mathbb{R})$.

We now wish to see whether this representation lifts to any representation of $\mathrm{sl}(4,\mathbb{R})$, upon inclusion of the weights in eq. (3.21). As mentioned above, the natural candidate is an 84-dimensional representation of $\mathrm{sl}(4,\mathbb{R})$ with Dynkin labels $[2,0,2]$. It is illuminating to first decompose it in terms of representations of $\mathrm{sl}(3,\mathbb{R})$,

$$\mathbf{84} = \mathbf{27} \oplus \mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{6} \oplus \overline{\mathbf{6}} \oplus \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1}, \tag{3.25}$$

or, in terms of Dynkin labels,

$$[2,0,2] = [2,2] + [2,1] + [1,2] + [2,0] + [0,2] + [1,1] + [1,0] + [0,1] + [0,0]. \tag{3.26}$$

We may view this decomposition as a *level decomposition* of the representation **84**, with the level ℓ being represented by the number of times the third simple root $\vec{\alpha}_3$ appears in each representation. From this point of view, and as we shall see in more detail shortly, the representations $\mathbf{27}, \mathbf{8}$ and $\mathbf{1}$ reside at $\ell = 0$, the representations $\mathbf{15}$ and $\mathbf{3}$ at $\ell = 1$, and the representation $\mathbf{6}$ at $\ell = 2$. The "barred" representations then reside at the associated negative levels. Knowing that we will only find the strictly positive weights in these representations, let us therefore start by listing these.

Firstly, we may neglect all representations at negative levels since these do not contain any positive weights. However, not all weights for $\ell \geq 0$ are positive. If we had decomposed the adjoint representation of $\mathrm{sl}(4,\mathbb{R})$ this problem would not have been present since all roots are either positive or negative, and hence all weights at positive level are positive and vice versa. In our case this is not true because for representations larger than the adjoint many weights are neither positive nor negative. It is furthermore important to realize that after dualisation it is the positive weights of $\mathrm{sl}(4,\mathbb{R})$ that we will obtain and not of $\mathrm{sl}(3,\mathbb{R})$. As can be seen in figure 1 the decomposition indeed includes weights which are negative weights of $\mathrm{sl}(3,\mathbb{R})$ but nevertheless positive weights of $\mathrm{sl}(4,\mathbb{R})$. An explicit counting reveals the following number of positive weights at each level (not counting weight multiplicities):

$$\begin{aligned} \ell = 0 &: 8, \\ \ell = 1 &: 8, \\ \ell = 2 &: 6. \end{aligned} \tag{3.27}$$

The eight weights at level zero are of course the positive weights of the **27** representation of $\mathrm{sl}(3,\mathbb{R})$ that we had before dualisation. In order to verify that we find all positive weights of **84** we must now check explicitly that after dualisation we get $8+6$ additional

Reps	ℓ	Positive Weights of $\mathrm{sl}(4,\mathbb{R})$
3	1	$\vec{\alpha}_3,\ \vec{\alpha}_2+\vec{\alpha}_3,\ \vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3$
15	1	$2\vec{\alpha}_2+\vec{\alpha}_3,\ \vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3,\ 2\vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3,$ $2\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3,\ (\vec{\alpha}_1+\vec{\alpha}_3)$
6	2	$2\vec{\alpha}_2+2\vec{\alpha}_3,\ \vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3,\ 2\vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3,$ $\vec{\alpha}_1+\vec{\alpha}_2+2\vec{\alpha}_3,\ 2\vec{\alpha}_3,\ \vec{\alpha}_2+2\vec{\alpha}_3$

Table 1: Positive weights at levels one and two.

positive weights. The total number of distinct weights of $\mathrm{sl}(4,\mathbb{R})$ that should appear in the Lagrangian after compactification and dualisation is thus 22.

The lifting from $\mathrm{sl}(3,\mathbb{R})$ to $\mathrm{sl}(4,\mathbb{R})$ is done by adding the third simple root $\vec{\alpha}_3 \equiv \vec{f}_3$, from eq. (3.21). The complete set of new weights arising from eq. (3.21) is then

$$
\begin{aligned}
\ell = 1 : \ & \vec{f}_1 = \vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3, && \vec{f}_2 = \vec{\alpha}_2+\vec{\alpha}_3,\\
& 2\vec{f}_1-\vec{f}_2 = 2\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3, && 2\vec{f}_2-\vec{f}_3 = 2\vec{\alpha}_2+\vec{\alpha}_3,\\
& 2\vec{f}_1-\vec{f}_3 = 2\vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3, && \vec{f}_1+\vec{f}_2-\vec{f}_3 = \vec{\alpha}_1+2\vec{\alpha}_2+\vec{\alpha}_3,\\
& (\vec{f}_1+\vec{f}_3-\vec{f}_2 = \vec{\alpha}_1+\vec{\alpha}_3), && \vec{f}_3 = \vec{\alpha}_3,\\
\\
\ell = 2 : \ & 2\vec{f}_1 = 2\vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3, && 2\vec{f}_2 = 2\vec{\alpha}_2+2\vec{\alpha}_3,\\
& 2\vec{f}_3 = 2\vec{\alpha}_3, && \vec{f}_1+\vec{f}_2 = \vec{\alpha}_1+2\vec{\alpha}_2+2\vec{\alpha}_3,\\
& \vec{f}_1+\vec{f}_3 = \vec{\alpha}_1+\vec{\alpha}_2+2\vec{\alpha}_3, && \vec{f}_2+\vec{f}_3 = \vec{\alpha}_2+2\vec{\alpha}_3.
\end{aligned}
\tag{3.28}
$$

In table 1 we indicate which representations these weights belong to and in figure 1 we give a graphical presentation of the level decomposition. The weight $\vec{\alpha}_1+\vec{\alpha}_3$ is put inside a parenthesis since terms giving this particular dilaton exponent in the Gauss-Bonnet combination are all absorbed into the equations of motion, and thus do not contribute according to our compactification procedure. However, generically it will contribute for a general second order curvature correction. We suspect the origin of this "missing" weight is connected to the mismatch in the multiplicity counting, which we will discuss briefly below. These results show that the Gauss-Bonnet term in $D = 6$ compactified on T^3 to three dimensions gives rise to strictly positive weights that can all be fit into the **84**-representation of $\mathrm{sl}(4,\mathbb{R})$.

Weight multiplicities. We have shown that the six-dimensional Gauss-Bonnet term compactified to three dimensions gives rise to positive weights of the **84**-representation of $\mathrm{sl}(4,\mathbb{R})$. However, we have not yet addressed the issue of weight multiplicities. It is not clear how to approach this problem. Naively, one might argue that if k distinct terms in the Lagrangian are multiplied by the same dilaton exponential, corresponding to some weight $\vec{\lambda}$, then this weight has multiplicity k. Unfortunately, this type of counting does not seem to work, one of the reasons being that the notion of distinctness is not clearly defined.

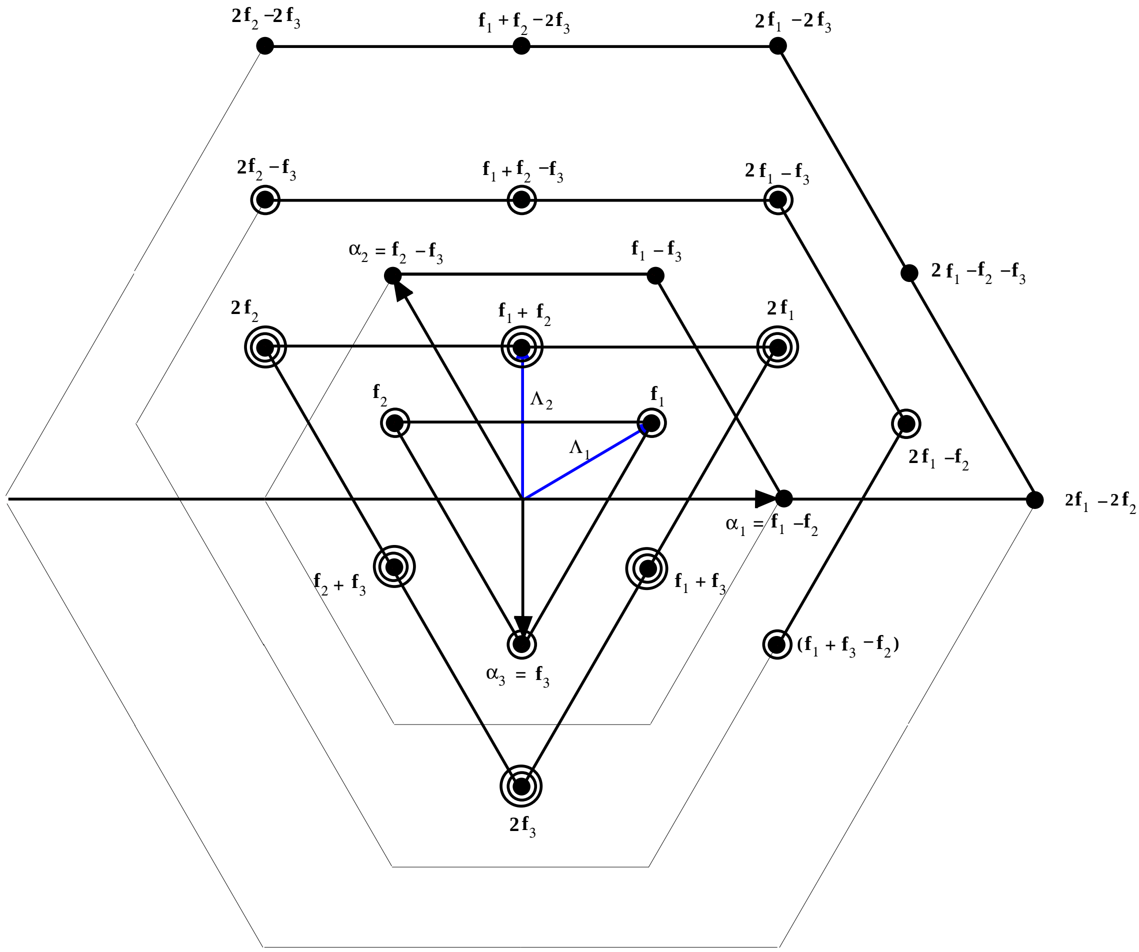

Figure 1: Graphical presentation of the representation structure of the compactified Gauss-Bonnet term. The black nodes arise from distinct dilaton exponents in the three-dimensional Lagrangian. The figure displays the level decomposition of the **84**-representation of $\mathrm{sl}(4,\mathbb{R})$ into representations of $\mathrm{sl}(3,\mathbb{R})$. Only positive levels are displayed. The black nodes correspond to positive weights of **84** of $\mathrm{sl}(4,\mathbb{R})$. Nodes with no rings represent the positive weights of the level zero representation **27**, nodes with one ring represent the positive weights of the level one representations **15** and **3**, while nodes with two rings represent the positive weights of the level two representation **6**. The shaded lines complete the representations with non-positive weights which are not displayed explicitly. The missing weight is put into a parenthesis.

Consider, for instance, the representations at $\ell = 1$. Both representations **15** and **3** contain the weights $\vec{f}_1$, $\vec{f}_2$ and $\vec{f}_3$. In **15** these have all multiplicity 2, while in **3** they have multiplicity 1. Thus, in total these weights have multiplicity 3 as weights of $\mathrm{sl}(3,\mathbb{R})$. Now, a detailed investigation reveals that the dilaton exponent $\vec{f}_a$ appears in the Gauss-Bonnet term accompanied with various different constraints on the index a, the no constraint case given in eq. (3.21) is merely the "most unconstrained" one. It can be easily shown that weights with lower value on index a have higher multiplicity. We therefore deduce that for all these weights there appears to be a mismatch in the multiplicity.

We suggest that the correct way to interpret this discrepancy in the weight multiplicities is as an indication of the need to introduce transforming automorphic forms in order to restore the $\mathrm{SL}(4,\mathbb{Z})$-invariance. This will be discussed more closely in section 4.

Including the Dilaton prefactor. We will now revisit the analysis from section 3.3, but here we include the contribution from the overall exponential factor $e^{-2\varphi}$ in the Lagrangian eq. (2.20). This factor arises as follows. The determinant of the D-dimensional vielbein is given by $\hat{e} = e^{D\varphi}\tilde{e}$, because of the Weyl-rescaling. Moreover, upon compactification the determinant of the rescaled vielbein splits according to $\tilde{e} = e\tilde{e}_{\text{int}}$, where e represents the external vielbein and $\tilde{e}_{\text{int}}$ the internal vielbein. The Weyl-rescaling is then chosen to be defined as $\tilde{e}_{\text{int}} = e^{-(D-2)\varphi}$. This represents the volume of the n-torus, upon which we perform the reduction. Thus, the overall scaling contribution from the measure is $e^{D\varphi}e^{-(D-2)\varphi} = e^{2\varphi}$. In addition, we have a factor of $e^{-4\varphi}$ from Weyl-rescaling the Gauss-Bonnet term (see eq. (A.20) and eq. (A.21)). This gives a total overall dilaton prefactor of $e^{-2\varphi}$, which, after inserting $\varphi = \frac{1}{6}\vec{g}\cdot\vec{\phi}$, becomes $e^{-\frac{1}{3}\vec{g}\cdot\vec{\phi}}$.

The importance of the volume factor for compactified higher derivative terms was emphasized in [7], using the argument that after dualisation this factor is no longer invariant under the extended symmetry group $\mathrm{SL}(n+1,\mathbb{R})$ and so must be included in the weight structure. We shall see that the inclusion of this factor drastically modifies the previously presented structure.

The fundamental weights of $\mathfrak{sl}(4,\mathbb{R})$. In order to perform this analysis, it is useful to first rewrite the simple roots and fundamental weights in a way which makes a comparison with [7] possible. We define arbitrary 3-vectors in $\mathbb{R}^3$ as follows

$$\hat{\vec{v}} = v_1\vec{\Lambda}_1 + v_2\vec{\Lambda}_2 + v_g\vec{g} = \big(\vec{v}, v_g\big) = \big(v_1, v_2, v_g\big), \tag{3.29}$$

where $\vec{\Lambda}_1$ and $\vec{\Lambda}_2$ are the fundamental weights of $\mathrm{sl}(3,\mathbb{R})$ and $\vec{g}$ is the basis vector taking us from the weight space $\mathbb{R}^2$ of $\mathrm{sl}(3,\mathbb{R})$ to the weight space $\mathbb{R}^3$ of $\mathrm{sl}(4,\mathbb{R})$. Note that

$$\vec{\Lambda}_1\cdot\vec{g} = \vec{\Lambda}_2\cdot\vec{g} = 0, \tag{3.30}$$

by virtue of eq. (3.7) and eq. (3.11), which implies

$$\hat{\vec{v}}\cdot\hat{\vec{u}} = \vec{v}\cdot\vec{u} + v_g u_g \vec{g}\cdot\vec{g}. \tag{3.31}$$

The scalar products may all be deduced using the orthonormal basis $\vec{e}_i$ of $\mathbb{R}^3$. Restricting to $D=6$ and $n=3$ gives

$$\vec{f}_a = \sqrt{2}\vec{e}_a + \frac{2}{9}\vec{g}, \tag{3.32}$$

and thus

$$\vec{\omega}_a = \vec{f}_a - \frac{4}{9}\vec{g}. \tag{3.33}$$

The relevant scalar products become

$$\begin{aligned}\vec{g}\cdot\vec{g} &= \frac{27}{2},\\ \vec{g}\cdot\vec{f}_a &= 6,\\ \vec{f}_a\cdot\vec{f}_b &= 2\delta_{ab}+2,\\ \vec{\omega}_a\cdot\vec{\omega}_b &= 2\delta_{ab}-\frac{2}{3}.\end{aligned}\tag{3.34}$$

The simple roots of $\mathrm{sl}(3,\mathbb{R})$ may now be written as

$$\begin{aligned}\hat{\vec{\alpha}}_1 &= \big(\vec{\alpha}_1,0\big) = \big(2,-1,0\big),\\ \hat{\vec{\alpha}}_2 &= \big(\vec{\alpha}_2,0\big) = \big(-1,2,0\big),\end{aligned}\tag{3.35}$$

and the third simple root becomes

$$\hat{\vec{\alpha}}_3 = \vec{f}_3 = \vec{\omega}_3 + \frac{4}{9}\vec{g} = -\vec{\Lambda}_2 + \frac{4}{9}\vec{g} = \left(0,-1,\frac{4}{9}\right).\tag{3.36}$$

In addition, the associated fundamental weights $\hat{\vec{\Lambda}}_i$, $i=1,2,3$, of $\mathrm{sl}(4,\mathbb{R})$, defined by

$$\hat{\vec{\alpha}}_i\cdot\hat{\vec{\Lambda}}_j = 2\delta_{ij},\tag{3.37}$$

become

$$\hat{\vec{\Lambda}}_1 = \left(1,0,\frac{1}{9}\right),\quad \hat{\vec{\Lambda}}_2 = \left(0,1,\frac{2}{9}\right),\quad \hat{\vec{\Lambda}}_3 = \left(0,0,\frac{1}{3}\right).\tag{3.38}$$

Let us check that these indeed correspond to the fundamental weights of $\mathrm{sl}(4,\mathbb{R})$, by computing the highest weight $2\hat{\vec{\Lambda}}_1+2\hat{\vec{\Lambda}}_3$ explicitly,

$$\begin{aligned}2\hat{\vec{\Lambda}}_1+2\hat{\vec{\Lambda}}_3 &= 2\vec{\Lambda}_1+\frac{2}{9}\vec{g}+\frac{2}{3}\vec{g}\\ &= 2\left(\vec{\omega}_1+\frac{4}{9}\vec{g}\right)\\ &= 2\vec{f}_1\\ &= 2\hat{\vec{\alpha}}_1+2\hat{\vec{\alpha}}_2+2\hat{\vec{\alpha}}_3.\end{aligned}\tag{3.39}$$

This result is consistent with being the highest weight of the **84** representation of $\mathrm{sl}(4,\mathbb{R})$ as can be seen in figure 1.

Dualisation and the overall Dilaton factor. Let us now include the dilaton prefactor in the analysis. In terms of $\mathrm{sl}(4,\mathbb{R})$-vectors the volume factor can be identified with a negative shift in $\hat{\vec{\Lambda}}_3$, i.e.,

$$e^{-\frac{1}{3}\vec{g}\cdot\vec{\phi}} = e^{-\hat{\vec{\Lambda}}_3\cdot\vec{\phi}}.\tag{3.40}$$

As already mentioned above, this factor is irrelevant before dualisation because $\vec{g}\cdot\vec{\phi}$ is invariant under $\mathrm{SL}(3,\mathbb{R})$. Thus, before dualisation the manifest $\mathrm{SL}(3,\mathbb{R})$-symmetry of the compactified Gauss-Bonnet term is associated with the **27**-representation of $\mathrm{sl}(3,\mathbb{R})$.

After dualisation, all the dilaton exponents in eq. (3.20) and eq. (3.21) become shifted by a factor of $-\hat{\vec{\Lambda}}_3$. In particular, the new highest weight is

$$\left(2\hat{\vec{\Lambda}}_1 + 2\hat{\vec{\Lambda}}_3\right) - \hat{\vec{\Lambda}}_3 = 2\hat{\vec{\Lambda}}_1 + \hat{\vec{\Lambda}}_3, \tag{3.41}$$

corresponding to the **36** representation of $\mathrm{sl}(4,\mathbb{R})$, with Dynkin labels $[2,0,1]$. This is consistent with the general result of [7] that a generic curvature correction to pure Einstein gravity of order $l/2$ should be associated with an $\mathrm{sl}(n+1,\mathbb{R})$-representation with highest weight $\frac{l}{2}\hat{\vec{\Lambda}}_1 + \hat{\vec{\Lambda}}_n$.

However, this is not the full story. A more careful examination in fact reveals that the **36** representation cannot incorporate all the dilaton exponents appearing in the Lagrangian, in contrast to the **84**-representation of figure 1. To see this, let us decompose **36** in terms of representations of $\mathrm{sl}(3,\mathbb{R})$. The result is:

$$\begin{aligned} \mathbf{36} &= \mathbf{15} \oplus \mathbf{8} \oplus \mathbf{6} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}, \\ [2,0,1] &= [2,1] + [1,1] + [2,0] + [1,0] + [0,1] + [0,0]. \end{aligned} \tag{3.42}$$

Comparing this with eq. (3.25), we see that the representations $\mathbf{27}, \bar{\mathbf{15}}$ and $\bar{\mathbf{6}}$ are no longer present. For the latter two this is not a problem since they were never present in the previous analysis. What happens is that the **6** of **84** gets shifted "downwards" and becomes the **6** of **36**. Similarly, the **15** and **3** of **84** become the **15** and **3** of **36**. This takes into account all the shifted dilaton exponents arising from the dualisation process. However, since there is not enough "room" for the **27** of $\mathrm{sl}(3,\mathbb{R})$ in eq. (3.42), some of the dilaton exponents (the ones corresponding to $2\vec{f}_2 - 2\vec{f}_3$, $\vec{f}_1 + \vec{f}_2 - 2\vec{f}_3$, $2\vec{f}_1 - 2\vec{f}_3, 2\vec{f}_1 - \vec{f}_2 - \vec{f}_3$ and $2\vec{f}_1 - 2\vec{f}_2$) arising from the pure $\tilde{P}$-terms remain outside of **36**. In fact, due to the shift of $-\hat{\vec{\Lambda}}_3$ these have now become *negative* weights of $\mathrm{sl}(4,\mathbb{R})$, because they are below the hyperplane defined by $\vec{g} \cdot \vec{x} = 0$. Although we know that these weights still correspond to positive weights of the **27** representation of $\mathrm{sl}(3,\mathbb{R})$, we are not able to determine which representation of $\mathrm{sl}(4,\mathbb{R})$ they belong to.

By a straightforward generalisation of this analysis to compactifications of quadratic curvature corrections from arbitrary dimensions D, we may conclude that the highest weight $2\hat{\vec{\Lambda}}_1 + \hat{\vec{\Lambda}}_n$, can never incorporate the dilaton exponents associated with the $[2,0,\ldots,0,2]$-representation of $\mathrm{sl}(n,\mathbb{R})$ before dualisation.

4. Discussion and conclusions

It is clear from the analysis in the previous section that the overall dilaton factor $e^{-\hat{\vec{\Lambda}}_3\cdot\vec{\phi}}$ (or, more generally, $e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}$) complicates the interpretation of the dilaton exponents in terms of $\mathrm{sl}(n+1,\mathbb{R})$-representations. A similar problem has arisen in attempts at incorporating the representation structure of the hyperbolic Kac-Moody algebra $E_{10(10)}$ into curvature corrections to string and M-theory [16, 17]. There it is the "lapse function" N which plays the role of the volume factor. Similarly to our findings, the work of [16, 17] reveals that curvature corrections to, e.g., eleven-dimensional supergravity, fit into *negative* weights of $E_{10(10)}$

if the contribution from the lapse function is included. In addition, there are indications that the relevant representations of $E_{10(10)}$ are so-called *non-integrable* representations, which are not well understood.

Given these considerations, it would be desirable to have an alternative interpretation of the results where one neglects the overall volume factor (or, in the $E_{10(10)}$-case, the lapse function) in the analysis of the weight structure.

First, what information does the weight structure contain? Apart from the overall dilaton factor, the reduction of any higher derivative term $\sim \mathcal{R}^p$ will give rise to terms with $\mathcal{P}^{2p}$ (and terms with more derivatives and fewer $\mathcal{P}$'s), where $\mathcal{P}$ represents any of the "building blocks" P, H and $\partial\phi$ (we suppress all 3-dimensional indices). The appearance of weights of $\mathrm{sl}(n+1,\mathbb{R})$ (without the uniform shift from the overall dilaton factor) reflects the fact that we use fields which are components of the symmetric part of the left-invariant Maurer-Cartan form $\mathcal{P}$ of $\mathrm{sl}(n+1,\mathbb{R})$. Moreover, the dilaton factor contains information about the number of such fields. A term $\mathcal{R}^{l/2}$ will generically give weights in the weight space of the representation $[l/2, 0, \ldots, 0, l/2]$ of $\mathrm{sl}(n+1,\mathbb{R})$, and fill out the positive part of this weight space.[7] This much is clear from the observation that the overall dilaton factor really is "overall".

The presence of the overall dilaton factor shifts this weight space uniformly in a negative direction. This shift happens to be by a vector in the weight lattice of $\mathrm{sl}(n+1,\mathbb{R})$ for any value of p. However, we emphasize that the dilaton exponents still lie in the weight space of the representation $[l/2, 0, \ldots, 0, l/2]$, albeit shifted "downwards". From this point of view, the weight space of the representation with the shifted highest weight of $[l/2, 0, \ldots, 0, l/2]$ as highest weight – for example, the representation $[2, 0, 1]$ in the case discussed above — does not contain all the weights that appear in the reduced Lagrangian, and therefore does not appear to be relevant.

4.1 An $\mathrm{SL}(n+1,\mathbb{Z})$-invariant effective action

Consider now the fact that it is really the discrete "U-duality" group $\mathrm{SL}(n+1,\mathbb{Z}) \subset \mathrm{SL}(n+1,\mathbb{R})$ which is expected to be a symmetry of the complete effective action. Therefore, the compactified action should be seen as a remnant of the full U-duality invariant action, arising from a "large volume expansion" of certain automorphic forms.

Schematically, a generic, quartic, scalar term in the action after compactification of the Gauss-Bonnet term is of the form

$$\int d^3x \sqrt{|g|} e^{-\hat{\vec{\Lambda}}_n \cdot \vec{\phi}} F(\mathcal{P}), \tag{4.1}$$

where $F(\mathcal{P})$ is a quartic polynomial in the components of the Maurer-Cartan form mentioned above. F will be invariant under $\mathrm{SO}(n)$ by construction, but generically not under $\mathrm{SO}(n+1)$.

To obtain an action which is a scalar under $\mathrm{SO}(n+1)$ we must first "lift" the result of the compactification to a globally $\mathrm{SL}(n+1,\mathbb{Z})$-invariant expression. This can be done

[7] We note that the representation structure encountered here is of the same type as for the lattice of BPS charges in string theory on T^n [28].

by replacing $e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}F(\mathcal{P})$ by a suitable automorphic form contracted with four $\mathcal{P}$'s:

$$\Psi_{I_1\ldots I_8}(X)\mathcal{P}^{I_1I_2}\mathcal{P}^{I_3I_4}\mathcal{P}^{I_5I_6}\mathcal{P}^{I_7I_8}, \tag{4.2}$$

where the I's are vector indices of $\mathrm{SO}(n+1)$. Here, $\Psi(X)$ is an automorphic form transforming in some representation of $\mathrm{SO}(n+1)$, and is constructed as an Eisenstein series, following, e.g., refs. [8, 9]. We must demand that when the large volume limit, $\hat{\vec{\Lambda}}_n\cdot\vec{\phi}\to-\infty$, is imposed, the leading behaviour is

$$\Psi_{I_1\ldots I_8}(X)\mathcal{P}^{I_1I_2}\mathcal{P}^{I_3I_4}\mathcal{P}^{I_5I_6}\mathcal{P}^{I_7I_8} \quad\longrightarrow\quad e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}F(\mathcal{P}). \tag{4.3}$$

This limit was taken explicitly in [8, 9]. This gives conditions on which irreducible $\mathrm{SO}(n{+}1)$ representations the automorphic forms transform under (from the tensor structure), as well as a single condition on the "weights" of the automorphic forms (from the matching of the overall dilaton factor). Automorphic forms exist for continuous values of the weight (unlike holomorphic Eisenstein series) above some minimal value derived from convergence of the Eisenstein series. It was proven in [9] that any $\mathrm{SO}(n)$-covariant tensor structure can be reproduced as the large volume limit of some automorphic form, and that the weight dictated by the overall dilaton factor is consistent with the convergence criterion.

Under the assumption that these arguments are valid, we may conclude that the representation theoretic structure of the dilaton exponents in the polynomial F should be analyzed without inclusion of the volume factor $e^{-\hat{\vec{\Lambda}}_n\cdot\vec{\phi}}$, and hence, for the Gauss-Bonnet term ($l=4$), it is the $[2,0,\ldots,0,2]$-representation which is the relevant one (in the sense above, that we are dealing with products of four Maurer-Cartan forms), and *not* the representation $[2,0,\ldots,0,1]$. Another indication for why the representation with highest weight $2\hat{\vec{\Lambda}}_1+\hat{\vec{\Lambda}}_n$ cannot be the relevant one is that it is not contained in the tensor product of the adjoint representation $[1,0,\ldots,0,1]$ of $\mathrm{sl}(n+1,\mathbb{R})$ with itself.

The present point of view also suggest a possible explanation for the discrepancy of the weight multiplicities observed in the previous section. In the complete $\mathrm{SL}(n{+}1,\mathbb{Z})$-invariant four-derivative effective action the multiplicities of the weights in the $[2,0,\ldots,0,2]$-representation necessarily match because the action is constructed directly from the $\mathrm{sl}(n+1,\mathbb{R})$-valued building block $\mathcal{P}$. When taking the large volume limit, eq. (4.3), a lot of information is lost (see, e.g., [9]) and it is therefore natural that the result of the compactification does not display the correct weight multiplicities. Thus, it is only after taking the non-perturbative completion, eq. (4.2), that we can expect to reproduce correctly the weight multiplicities of the representation $[2,0,\ldots,0,2]$.

4.2 Algebraic constraints on curvature corrections

Our results have additional implications for the interpretations of the weight structure laid forward in [7]. In the analysis of the compactification of eleven-dimensional supergravity to three dimensions these authors include the volume factor when investigating the weight structure of $E_{8(8)}$. This implies that an arbitrary weight for the $l/2$:th order correction terms contains a factor of $(\frac{1}{3}-\frac{l}{6})\hat{\vec{\Lambda}}_8$. In our example above this precisely corresponds to the volume factor $\hat{\tilde{\Lambda}}_n$. Including this factor and demanding that all dilaton exponents should be on the weight lattice of $E_{8(8)}$ gives the constraint

$$\frac{1}{3} - \frac{l}{6} \in \mathbb{Z} \quad \Longleftrightarrow \quad l = 6k + 2, \quad (k = 0, 1, 2, \dots). \tag{4.4}$$

This implies that these can only be on the weight lattice of $E_{8(8)}$ if the orders of the curvature correction are the celebrated powers $\frac{l}{2} = 3k + 1$, $k = 0, 1, 2, \dots$. However, if our interpretation is correct, the volume factor should be left outside of the representation structure and so this argument about the restrictions on l does not seem to be applicable from a purely mathematical point of view, since also intermediate values can be reproduced by automorphic forms with some (continuous) weight.[8]

This, of course, does not mean that the result itself is incorrect (it is well known, e.g., that the first higher-derivative correction allowed by supersymmetry is of order $\mathcal{R}^4$, as is the first correction obtained by superstring calculations), only that the arguments used in order to reach it have to be refined. In order to obtain the result in the present context, one would need information restricting the weights of the automorphic forms that may enter to some discrete values. Real automorphic forms defined by Eisenstein series, unlike the holomorphic ones of $\mathrm{SL}(2, \mathbb{R})$ (or $\mathrm{Sp}(2n)$ in general), are defined for continuous values of the weight, bounded from below only by the convergence of the series. When one-loop calculations in eleven-dimensional supergravity have been used to derive automorphic forms occurring in $d = 9$ [12], it is clear how well-defined values of the weights arise. The corresponding picture for compactification to lower dimensions is less clear, due to the presence of membrane and 5-brane instantons [28, 29], but there is no doubt a corresponding mechanism at play, although we lack enough insight into the microscopic degrees of freedom to make a clear statement about it.

We suspect that a reasoning along similar lines may be used for the case of $E_{10(10)}$, and that it may again lead to the conclusion that the shifted highest weight should not be interpreted as the highest weight of a new (non-integrable) representation. Instead, it may be possible to deal with automorphic forms transforming in some integrable representations of the maximal compact subgroup of $E_{10(10)}$.

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[8]The fact that $E_{8(8)}$-invariant terms which do not arise from the compactification of $\mathcal{R}^{3k+1}$ curvature corrections can exist in $D = 3$ follows also from the work of [8], which however emphasizes a different role of the dilaton pre-factors compared to the one suggested here. We thank the authors of [8] for correspondence on this issue.

A. Squared curvature terms

Here we present all the detailed computations of the compactification.

A.1 Weyl-rescaling

Weyl-rescaling the D-dimensional metric by a factor $e^{2\varphi}$:

$$\hat{g}_{MN} = e^{2\varphi}\tilde{g}_{MN}, \tag{A.1}$$

yields the rescaled Riemann tensor

$$\begin{aligned}\hat{R}_{ABCD} &= e^{-2\varphi}\big[\tilde{R}_{ABCD} - 2\big(\eta_{[A|C|}\tilde{\nabla}_{B]}\tilde{\partial}_D\varphi - \eta_{[A|D|}\tilde{\nabla}_{B]}\tilde{\partial}_C\varphi\big)\\ &\qquad +2\big(\eta_{[A|C|}\tilde{\partial}_{B]}\varphi\tilde{\partial}_D\varphi - \eta_{[A|D|}\tilde{\partial}_{B]}\varphi\tilde{\partial}_C\varphi\big) - 2\eta_{[A|C|}\eta_{B]D}(\tilde{\partial}\varphi)^2\big],\end{aligned} \tag{A.2}$$

Ricci tensor

$$\hat{R}_{AB} = e^{-2\varphi}\big[\tilde{R}_{AB} - \eta_{AB}\tilde{\Box}\varphi - (D-2)\tilde{\nabla}_A\tilde{\partial}_B\varphi + (D-2)\tilde{\partial}_A\varphi\tilde{\partial}_B\varphi - (D-2)\eta_{AB}(\tilde{\partial}\varphi)^2\big], \tag{A.3}$$

and curvature scalar

$$\hat{R} = e^{-2\varphi}\big[\tilde{R} - (D-1)(D-2)(\tilde{\partial}\varphi)^2 - 2(D-1)\tilde{\Box}\varphi\big]. \tag{A.4}$$

Squaring the curvature terms we find

$$\begin{aligned}(\hat{R}_{ABCD})^2 &= e^{-4\varphi}\Bigg[(\tilde{R}_{ABCD})^2 + 8\bigg(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\bigg)\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi - 8\tilde{R}_{AB}\tilde{\nabla}^A\tilde{\partial}^B\varphi + 4(\tilde{\Box}\varphi)^2\\ &\qquad +4(D-2)(\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\nabla}^A\tilde{\partial}^B\varphi) + 8(D-2)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi\\ &\qquad -8(D-2)\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi\tilde{\nabla}_A\tilde{\partial}_B\varphi + 2(D-1)(D-2)(\tilde{\partial}\varphi)^2(\tilde{\partial}\varphi)^2\Bigg],\end{aligned} \tag{A.5}$$

$$\begin{aligned}(\hat{R}_{AB})^2 &= e^{-4\varphi}\big[(\tilde{R}_{AB})^2 - 2\tilde{R}\tilde{\Box}\varphi - 2(D-2)\tilde{R}_{AB}\tilde{\nabla}^A\tilde{\partial}^B\varphi + (3D-4)(\tilde{\Box}\varphi)^2\\ &\qquad +2(D-2)\big(\tilde{R}_{AB} - \eta_{AB}\tilde{R}\big)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi) + (D-2)^2(\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\nabla}^A\tilde{\partial}^B\varphi)\\ &\qquad +(D-1)(D-2)^2(\tilde{\partial}\varphi)^4 + 2(D-2)(2D-3)(\tilde{\Box}\varphi)(\tilde{\partial}\varphi)^2\\ &\qquad -2(D-2)^2(\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi)\big],\end{aligned} \tag{A.6}$$

$$\begin{aligned}\hat{R}^2 &= e^{-4\varphi}\big[\tilde{R}^2 - 4(D-1)\tilde{R}\tilde{\Box}\varphi - 2(D-1)(D-2)\tilde{R}(\tilde{\partial}\varphi)^2 + 4(D-1)^2(\tilde{\Box}\varphi)^2\\ &\qquad +4(D-1)^2(D-2)(\tilde{\Box}\varphi)(\tilde{\partial}\varphi)^2 + (D-1)^2(D-2)^2(\tilde{\partial}\varphi)^4\big].\end{aligned} \tag{A.7}$$

Combining these, the Gauss-Bonnet combination can be written as

$$\begin{aligned}\hat{R}^2_{\rm GB} &= e^{-4\varphi}\Bigg\{\tilde{R}^2_{\rm GB} + (D-3)\Bigg[8\bigg(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\bigg)\tilde{\nabla}^A\tilde{\partial}^B\varphi - 8\tilde{R}_{AB}\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi\\ &\qquad -2(D-4)\tilde{R}(\tilde{\partial}\varphi)^2 + 4(D-2)(D-3)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi + 8(D-2)(\tilde{\nabla}_A\tilde{\partial}_B\varphi)\tilde{\partial}^A\varphi\tilde{\partial}^B\varphi\\ &\qquad +4(D-2)[(\tilde{\Box}\varphi)^2 - (\tilde{\nabla}_A\tilde{\partial}_B\varphi)(\tilde{\nabla}^A\tilde{\partial}^B\varphi)] + (D-1)(D-2)(D-4)(\tilde{\partial}\varphi)^4\Bigg]\Bigg\}.\end{aligned} \tag{A.8}$$

The Gauss-Bonnet Lagrangian, including the measure $\hat{e} = e^{D\varphi}\tilde{e}$, can now be conveniently grouped in terms of equations of motion and total derivatives. This is achieved using integrations by parts, where no explicit appearance of $\tilde{\nabla}_{(A}\tilde{\partial}_{B)}\varphi$ is required. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm GB} = \tilde{e}e^{(D-4)\varphi}\Big\{&\tilde{R}^2_{\rm GB} - (D-3)(D-4)\Big[2(D-2)(\tilde{\partial}\varphi)^2\tilde{\Box}\varphi + (D-2)(D-3)(\tilde{\partial}\varphi)^4 \\ &+4\Big(\tilde{R}_{AB} - \frac{1}{2}\eta_{AB}\tilde{R}\Big)(\tilde{\partial}^A\varphi)(\tilde{\partial}^B\varphi)\Big]\Big\} \\ +2(D-3)\tilde{e}\tilde{\nabla}_A\Big\{&e^{(D-4)\varphi}\Big[(D-2)^2(\tilde{\partial}\varphi)^2\tilde{\partial}^A\varphi + 2(D-2)(\tilde{\Box}\varphi)\tilde{\partial}^A\varphi \\ &-(D-2)\tilde{\partial}^A(\tilde{\partial}\varphi)^2 + 4\Big(\tilde{R}^{AB} - \frac{1}{2}\eta^{AB}\tilde{R}\Big)\tilde{\partial}_B\varphi\Big]\Big\}. \end{aligned} \tag{A.9}$$

Notice that the total derivative terms in this expression will remain total derivatives after the compactification as well.

A.2 Compactification

In compactification of gravity from D to $d = (D-n)$ dimensions the vielbein one-form is given by

$$\tilde{e}^A = (\tilde{e}^\alpha, \tilde{e}^a) = \big(e^\alpha, [dx^m + \mathcal{A}^m_{(1)}]\tilde{e}_m{}^a\big), \tag{A.10}$$

with the determinant denoted by $\tilde{e} = e\tilde{e}_{\rm int}$. Dropping all dependence on the torus coordinates, i.e., $\tilde{d} = d = dx^\mu\partial_\mu$, the compactified spin connection one-form is found to be

$$\begin{aligned}\tilde{\omega}^\alpha{}_\beta &= \omega^\alpha{}_\beta - \frac{1}{2}\tilde{e}^c\tilde{F}_c{}^\alpha{}_\beta, \\ \tilde{\omega}^\alpha{}_b &= \frac{1}{2}e^\gamma\tilde{F}_{b\gamma}{}^\alpha - \tilde{e}^c\tilde{P}^\alpha{}_{cb}, \\ \tilde{\omega}^a{}_b &= e^\gamma Q_\gamma{}^a{}_b, \end{aligned} \tag{A.11}$$

where $\tilde{P}_\alpha{}^{bc} = \tilde{e}^{m(b}\partial_\alpha\tilde{e}_m{}^{c)}$, $Q_\alpha{}^{bc} = \tilde{e}^{m[b}\partial_\alpha\tilde{e}_m{}^{c]}$ and $\tilde{F}^a{}_{\beta\gamma} = 2\tilde{e}_m{}^a e^\mu{}_{[\beta}e^\nu{}_{\gamma]}\partial_\mu\mathcal{A}^m_\nu$.

Using the spin connection it is now straightforward to compute the compactified Riemann tensor

$$\begin{aligned}\tilde{R}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} - \frac{1}{2}\big(\tilde{F}_{c\alpha[\gamma}\tilde{F}^c{}_{|\beta|\delta]} + \tilde{F}_{c\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\big), \\ \tilde{R}_{\alpha\beta\gamma d} &= D_{[\alpha}\tilde{F}_{|d|\beta]\gamma} - \tilde{F}^c{}_{\alpha\beta}\tilde{P}_{\gamma cd}, \\ \tilde{R}_{\alpha\beta cd} &= \frac{1}{2}\tilde{F}_{[c|\alpha|}{}^\gamma\tilde{F}_{d]\gamma\beta} - 2\tilde{P}_\alpha{}^e{}_{[c}\tilde{P}_{|\beta|d]e}, \\ \tilde{R}_{\alpha b\gamma d} &= -D_\alpha\tilde{P}_{\gamma bd} - \tilde{P}_{\alpha b}{}^e\tilde{P}_{\gamma de} + \frac{1}{4}\tilde{F}_{b\gamma\epsilon}\tilde{F}_{d\alpha}{}^\epsilon, \\ \tilde{R}_{ab\gamma d} &= \tilde{F}_{[a|\gamma\epsilon|}\tilde{P}^\epsilon{}_{b]d}, \\ \tilde{R}_{abcd} &= -2\tilde{P}_{\epsilon a[c}\tilde{P}^\epsilon{}_{|b|d]}, \end{aligned} \tag{A.12}$$

which contracted yields the Ricci tensor

$$\begin{aligned}
\tilde{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}\tilde{F}_{c\epsilon\alpha}\tilde{F}^{c\epsilon}{}_{\beta} - \tilde{P}_{\alpha cd}\tilde{P}_{\beta}{}^{cd} - \mathrm{tr}(D_\alpha\tilde{P}_\beta),\\
\tilde{R}_{\alpha b} &= \frac{1}{2}\big(D_\epsilon\tilde{F}_{b\alpha}{}^{\epsilon} + \tilde{F}_{c\alpha\delta}\tilde{P}^{\delta c}{}_{b} + \tilde{F}_{b\alpha\epsilon}\mathrm{tr}\tilde{P}^{\epsilon}\big),\\
\tilde{R}_{ab} &= -D^{\epsilon}\tilde{P}_{\epsilon ab} - \tilde{P}_{\epsilon ab}\mathrm{tr}\tilde{P}^{\epsilon} + \frac{1}{4}\tilde{F}_{a\gamma\delta}\tilde{F}_{b}{}^{\gamma\delta},
\end{aligned} \tag{A.13}$$

and the curvature scalar

$$\tilde{R} = R - \frac{1}{4}\tilde{F}^2 - \tilde{P}^2 - (\mathrm{tr}\tilde{P})^2 - 2\mathrm{tr}(D_\epsilon\tilde{P}^{\epsilon}). \tag{A.14}$$

The trace is always taken over the internal indices, also $\tilde{F}^2 \equiv \tilde{F}_{a\beta\gamma}\tilde{F}^{a\beta\gamma}$ and $\tilde{P}^2 \equiv \tilde{P}_{\alpha bc}\tilde{P}^{\alpha bc}$. The covariant derivative D is defined as $D \equiv \nabla + Q \equiv \partial + \omega + Q$, where $\omega_{\alpha\beta\gamma}$ is the spacetime spin connection and $Q_{\alpha bc}$ can be thought of as a gauge connection for the SO(n)-symmetry.

Squaring the curvature tensor components we find:

$$\begin{aligned}
(\tilde{R}_{\alpha\beta\gamma\delta})^2 &= R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{3}{2}R_{\alpha\beta\gamma\delta}\tilde{F}_{e}{}^{\alpha\beta}\tilde{F}^{e\gamma\delta} + \frac{3}{8}\tilde{F}_{c\alpha\beta}\tilde{F}_{d}{}^{\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}^{d\gamma\delta}\\
&\quad + \frac{3}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}_{d}{}^{\alpha\gamma}\tilde{F}^{d\beta\delta},\\
(\tilde{R}_{\alpha\beta\gamma d})^2 &= (D_{[\alpha}\tilde{F}_{|d|\beta]\gamma})(D^{\alpha}\tilde{F}^{d\beta\gamma}) - 2(D_\alpha\tilde{F}_{d\beta\gamma})\tilde{F}^{c\alpha\beta}\tilde{P}^{\gamma d}{}_{c} + \tilde{F}^{a}{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_{c}{}^{\gamma\delta},\\
(\tilde{R}_{\alpha\beta cd})^2 &= 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}^{\alpha}\tilde{P}_\beta\tilde{P}^{\beta}) - 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^{\alpha}\tilde{P}^{\beta}) + \frac{1}{8}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta}\\
&\quad - \frac{1}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}_{d}{}^{\alpha\gamma}\tilde{F}^{d\beta\delta} - \tilde{F}^{c}{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_{d}{}^{\beta\delta} + \tilde{F}^{c}{}_{\beta\gamma}\tilde{P}^{\gamma}{}_{ce}\tilde{P}_{\delta}{}^{ed}\tilde{F}_{d}{}^{\beta\delta},\\
(\tilde{R}_{\alpha b\gamma d})^2 &= (D_\alpha\tilde{P}_{\gamma bd})(D^{\alpha}\tilde{P}^{\gamma bd}) + 2(D_\alpha\tilde{P}_{\gamma bd})\tilde{P}^{\alpha be}\tilde{P}^{\gamma d}{}_{e} - \frac{1}{2}(D_\alpha\tilde{P}_{\gamma bd})\tilde{F}^{b\alpha\epsilon}\tilde{F}^{d\gamma}{}_{\epsilon}\\
&\quad + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}^{\alpha}\tilde{P}_\gamma\tilde{P}^{\gamma}) + \frac{1}{16}\tilde{F}_{c\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}_{d}{}^{\alpha\gamma}\tilde{F}^{d\beta\delta} - \frac{1}{2}\tilde{F}^{b}{}_{\beta\gamma}\tilde{P}_{\delta be}\tilde{P}^{\gamma ed}\tilde{F}_{d}{}^{\beta\delta},\\
(\tilde{R}_{ab\gamma d})^2 &= \frac{1}{2}\big[\tilde{F}_{c}{}^{\alpha}{}_{\delta}\tilde{F}^{c\beta\delta}\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta) - \tilde{F}^{a}{}_{\beta\gamma}\tilde{P}_{\delta ad}\tilde{P}^{\gamma db}\tilde{F}_{b}{}^{\beta\delta}\big],\\
(\tilde{R}_{abcd})^2 &= 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta}) - 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^{\alpha}\tilde{P}^{\beta}).
\end{aligned} \tag{A.15}$$

The compactified Ricci tensor and curvature scalar squared are

$$\begin{aligned}
(\tilde{R}_{\alpha\beta})^2 &= R_{\alpha\beta}R^{\alpha\beta} - R_{\alpha\beta}\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} - 2R_{\alpha\beta}\mathrm{tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta}) - 2R_{\alpha\beta}\mathrm{tr}(D^{\alpha}\tilde{P}^{\beta})\\
&\quad + \mathrm{tr}(D_\alpha\tilde{P}_\beta)\mathrm{tr}(D^{\alpha}\tilde{P}^{\beta}) + 2\mathrm{tr}(D_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta}) + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta})\\
&\quad + \mathrm{tr}(D_\alpha\tilde{P}_\beta)\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} + \frac{1}{4}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta},\\
(\tilde{R}_{\alpha b})^2 &= \frac{1}{4}\big[(D_\gamma\tilde{F}_{b\alpha}{}^{\gamma})(D_\delta\tilde{F}^{b\alpha\delta}) + 2(D_\gamma\tilde{F}_{b\alpha}{}^{\gamma})\tilde{F}^{b\alpha\beta}\mathrm{tr}\tilde{P}_\beta + 2(D_\gamma\tilde{F}_{c\alpha}{}^{\gamma})\tilde{P}_{\beta}{}^{cd}\tilde{F}_{d}{}^{\alpha\beta}\\
&\quad + \tilde{F}^{c}{}_{\beta\gamma}\tilde{P}^{\gamma}{}_{ce}\tilde{P}_{\delta}{}^{ed}\tilde{F}_{d}{}^{\beta\delta} + 2\tilde{F}_{b\alpha\gamma}\tilde{P}_{\beta}{}^{bc}\tilde{F}_{c}{}^{\alpha\beta}\mathrm{tr}\tilde{P}^{\gamma} + \tilde{F}_{b\alpha\gamma}\tilde{F}^{b\alpha}{}_{\delta}(\mathrm{tr}\tilde{P}^{\gamma})(\mathrm{tr}\tilde{P}^{\delta})\big],\\
(\tilde{R}_{ab})^2 &= \mathrm{tr}(D_\alpha\tilde{P}^{\alpha}D_\beta\tilde{P}^{\beta}) + 2(D_\alpha\tilde{P}^{\alpha bc})\tilde{P}_{\beta bc}\mathrm{tr}\tilde{P}^{\beta} - \frac{1}{2}(D_\alpha\tilde{P}^{\alpha bc})\tilde{F}_{b\gamma\delta}\tilde{F}_{c}{}^{\gamma\delta}\\
&\quad + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)(\mathrm{tr}\tilde{P}^{\alpha})(\mathrm{tr}\tilde{P}^{\beta}) - \frac{1}{2}\tilde{F}_{b\alpha\beta}\tilde{P}_{\gamma}{}^{bc}\tilde{F}_{c}{}^{\alpha\beta}\mathrm{tr}\tilde{P}^{\gamma} + \frac{1}{16}\tilde{F}_{c\alpha\beta}\tilde{F}_{d}{}^{\alpha\beta}\tilde{F}^{c}{}_{\gamma\delta}\tilde{F}^{d\gamma\delta},
\end{aligned} \tag{A.16}$$

and

$$
\begin{aligned}
\tilde{R}^2 = {} & R^2 - \frac{1}{2}R\tilde{F}^2 - 4R\mathrm{tr}(D_\alpha\tilde{P}^\alpha) - 2R\tilde{P}^2 - 2R(\mathrm{tr}\tilde{P})^2 + \frac{1}{16}(\tilde{F}^2)^2 + \tilde{F}^2\mathrm{tr}(D_\alpha\tilde{P}^\alpha) \\
& + \frac{1}{2}\tilde{F}^2\tilde{P}^2 + \frac{1}{2}\tilde{F}^2(\mathrm{tr}\tilde{P})^2 + 4[\mathrm{tr}(D_\alpha\tilde{P}^\alpha)]^2 + 4\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\tilde{P}^2 + 4\mathrm{tr}(D_\alpha\tilde{P}^\alpha)(\mathrm{tr}\tilde{P})^2 \\
& + (\tilde{P}^2)^2 + 2\tilde{P}^2(\mathrm{tr}\tilde{P})^2 + ((\mathrm{tr}\tilde{P})^2)^2. \qquad \text{(A.17)}
\end{aligned}
$$

Choosing a basis where all explicit derivatives appearing are either divergences or total derivatives, we can rewrite three of the quadratic curvature components as

$$
\begin{aligned}
(\tilde{R}_{\alpha\beta\gamma d})^2 = {} & \frac{1}{2}\big[R_{\alpha\beta\gamma\delta}\tilde{F}_e{}^{\alpha\gamma}\tilde{F}^{e\beta\delta} - R_{\alpha\beta}\tilde{F}_{e\gamma}{}^{\alpha}\tilde{F}^{e\gamma\beta} + (D_\gamma\tilde{F}_{b\alpha}{}^{\gamma})(D_\delta\tilde{F}^{b\alpha\delta}) \\
& - \tilde{F}_{c\alpha\beta}(D_\gamma\tilde{P}^{\gamma cd})\tilde{F}_d{}^{\alpha\beta} + \tilde{F}^a{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_c{}^{\gamma\delta} - \tilde{F}^c{}_{\beta\gamma}\tilde{P}^\gamma{}_{ce}\tilde{P}_\delta{}^{ed}\tilde{F}_d{}^{\beta\delta} \\
& + 3\tilde{F}^c{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_d{}^{\beta\delta}\big] + \frac{1}{2}\nabla_\alpha\big[(D_\gamma\tilde{F}_{e\beta}{}^{\alpha})\tilde{F}^{e\beta\gamma} - (D_\gamma\tilde{F}_{e\beta}{}^{\gamma})\tilde{F}^{e\beta\alpha} \\
& + \tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}\big], \\
(\tilde{R}_{\alpha b\gamma d})^2 = {} & \frac{1}{2}(D_\alpha\tilde{F}_{c\beta}{}^{\alpha})\tilde{P}_\gamma{}^{cd}\tilde{F}_d{}^{\beta\gamma} - \frac{1}{8}\tilde{F}_{c\alpha\beta}(D_\gamma\tilde{P}^{\gamma cd})\tilde{F}_d{}^{\alpha\beta} + \mathrm{tr}[(D_\alpha\tilde{P}^\alpha)(D_\beta\tilde{P}^\beta)] \\
& - \mathrm{tr}[(D_\alpha\tilde{P}^\alpha)\tilde{P}_\beta\tilde{P}^\beta] - R_{\alpha\beta}\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) + \frac{1}{16}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta} \\
& - \frac{1}{4}\tilde{F}^a{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_c{}^{\gamma\delta} + 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta) - \mathrm{tr}(\tilde{P}_\alpha\tilde{P}^\alpha\tilde{P}_\beta\tilde{P}^\beta) + \nabla_\alpha\Big[\mathrm{tr}(\tilde{P}^\alpha\tilde{P}_\beta\tilde{P}^\beta) \\
& + \mathrm{tr}(\tilde{P}_\beta D^\beta\tilde{P}^\alpha - \tilde{P}^\alpha D^\beta\tilde{P}_\beta) + \frac{1}{8}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma} - \frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\gamma cd}\tilde{F}_d{}^{\beta\alpha}\Big], \\
(\tilde{R}_{\alpha\beta})^2 = {} & R_{\alpha\beta}R^{\alpha\beta} - R_{\alpha\beta}\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta} - 2R_{\alpha\beta}\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - R_{\alpha\beta}\mathrm{tr}\tilde{P}^\alpha\mathrm{tr}\tilde{P}^\beta - R\mathrm{tr}(D_\alpha\tilde{P}^\alpha) \\
& - (D_\alpha\tilde{F}_{c\delta}{}^{\alpha})\tilde{F}^{c\delta\beta}\mathrm{tr}\tilde{P}_\beta - 2\mathrm{tr}[(D_\alpha\tilde{P}^\alpha)\tilde{P}^\beta]\mathrm{tr}\tilde{P}_\beta + \mathrm{tr}(D_\alpha\tilde{P}^\alpha)\mathrm{tr}(D_\beta\tilde{P}^\beta) \\
& + \frac{1}{4}\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\tilde{F}^2 + \mathrm{tr}(D_\alpha\tilde{P}^\alpha)\mathrm{tr}(\tilde{P}_\beta\tilde{P}^\beta) + \frac{1}{4}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta} \\
& + \frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}\mathrm{tr}\tilde{P}_\alpha - \tilde{F}_{c\beta\gamma}\tilde{P}^{\gamma cd}\tilde{F}_d{}^{\beta\alpha}\mathrm{tr}\tilde{P}_\alpha + \tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_{\beta}\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) \\
& + \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) + \nabla_\alpha\big[\mathrm{tr}(D^\alpha\tilde{P}_\beta)\mathrm{tr}\tilde{P}^\beta - \mathrm{tr}(D^\beta\tilde{P}_\beta)\mathrm{tr}\tilde{P}^\alpha - \frac{1}{4}\tilde{F}^2\mathrm{tr}\tilde{P}^\alpha \\
& - 2(R^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}R)\mathrm{tr}\tilde{P}_\beta + \tilde{F}^{c\delta\alpha}\tilde{F}_{c\delta}{}^{\beta}\mathrm{tr}\tilde{P}_\beta + 2\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta)\mathrm{tr}\tilde{P}_\beta - \tilde{P}^2\mathrm{tr}\tilde{P}^\alpha\big]. \qquad \text{(A.18)}
\end{aligned}
$$

This choice of basis is only possible for the curvature terms squared; for general powers of the Riemann tensor no such basis exists. The square of the uncompactified curvature terms can now be written as a sum of the quadratic compactified curvature components

$$
\begin{aligned}
(\tilde{R}_{ABCD})^2 = {} & \tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta} + 4\tilde{R}_{\alpha\beta\gamma d}\tilde{R}^{\alpha\beta\gamma d} + 2\tilde{R}_{\alpha\beta cd}\tilde{R}^{\alpha\beta cd} + 4\tilde{R}_{\alpha b\gamma d}\tilde{R}^{\alpha b\gamma d} \\
& + 4\tilde{R}_{ab\gamma d}\tilde{R}^{ab\gamma d} + \tilde{R}_{abcd}\tilde{R}^{abcd}, \\
(\tilde{R}_{AB})^2 = {} & \tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta} + 2\tilde{R}_{\alpha b}\tilde{R}^{\alpha b} + \tilde{R}_{ab}\tilde{R}^{ab}. \qquad \text{(A.19)}
\end{aligned}
$$

Since the total volume measure is $\hat{e} = e^{D\varphi}e\tilde{e}_{\mathrm{int}}$, the factor $e^{D\varphi}\tilde{e}_{\mathrm{int}}$ has to be moved inside the total derivatives using integration by parts. The Riemann tensor squared will then be given by

$$
\begin{aligned}
&\hat{e}e^{-4\varphi}\big(\tilde{R}_{ABCD}\big)^2\\
&\quad = ee^{(D-4)\varphi}\tilde{e}_{\rm int}\Big\{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}-\frac{1}{2}R_{\alpha\beta\gamma\delta}\tilde{F}_e{}^{\alpha\beta}\tilde{F}^{e\gamma\delta}-2R_{\alpha\beta}\big[\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta}+2{\rm tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta})\big]\\
&\qquad +2D_{\alpha}\tilde{F}_{c\delta}{}^{\alpha}\big[D_{\beta}\tilde{F}^{c\delta\beta}+\tilde{P}_{\beta}{}^{cd}\tilde{F}_d{}^{\delta\beta}+{\rm tr}\tilde{P}_{\beta}\tilde{F}^{c\delta\beta}+(D-4)\partial_{\beta}\varphi\tilde{F}^{c\delta\beta}\big]+4{\rm tr}(D_{\alpha}\tilde{P}^{\alpha}D_{\beta}\tilde{P}^{\beta})\\
&\qquad -4{\rm tr}[(D_{\alpha}\tilde{P}^{\alpha})\tilde{P}_{\beta}\tilde{P}^{\beta}]-\frac{5}{2}\tilde{F}_{c\alpha\beta}(D_{\gamma}\tilde{P}^{\gamma cd})\tilde{F}_d{}^{\alpha\beta}+4{\rm tr}(\tilde{P}_{\alpha}D_{\beta}\tilde{P}^{\beta})\big[{\rm tr}\tilde{P}^{\alpha}+(D-4)\partial^{\alpha}\varphi\big]\\
&\qquad +\frac{1}{2}{\rm tr}(D_{\alpha}\tilde{P}^{\alpha})\big[\tilde{F}^2+4\tilde{P}^2\big]+\frac{3}{8}\tilde{F}_{c\alpha\beta}\tilde{F}_d{}^{\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}^{d\gamma\delta}+\frac{1}{2}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta}\\
&\qquad +\frac{1}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}_d{}^{\alpha\gamma}\tilde{F}^{d\beta\delta}+2\tilde{F}^c{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_d{}^{\beta\delta}+\tilde{F}^a{}_{\gamma\delta}\tilde{P}_{\epsilon ab}\tilde{P}^{\epsilon bc}\tilde{F}_c{}^{\gamma\delta}+2{\rm tr}(\tilde{P}_{\alpha}\tilde{P}_{\beta}\tilde{P}^{\alpha}\tilde{P}^{\beta})\\
&\qquad +2{\rm tr}(\tilde{P}_{\alpha}\tilde{P}_{\beta})\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta}-\frac{3}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}\big[{\rm tr}\tilde{P}_{\alpha}+(D-4)\partial_{\alpha}\varphi\big]+2{\rm tr}(\tilde{P}_{\alpha}\tilde{P}_{\beta}){\rm tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta})\\
&\qquad +\frac{1}{2}\big(\tilde{F}^2+4\tilde{P}^2\big)\big[({\rm tr}\tilde{P})^2+2(D-4){\rm tr}\tilde{P}_{\alpha}\partial^{\alpha}\varphi+(D-4)^2(\partial\varphi)^2+(D-4)\Box\varphi\big]\\
&\qquad -4{\rm tr}(\tilde{P}_{\alpha}\tilde{P}^{\alpha}\tilde{P}_{\beta})\big[{\rm tr}\tilde{P}^{\beta}+(D-4)\partial^{\beta}\varphi\big]\big\}+e\nabla_{\alpha}\big\{e^{(D-4)\varphi}\tilde{e}_{\rm int}\big[-2(D_{\beta}\tilde{F}_{c\delta}{}^{\beta})\tilde{F}^{c\delta\alpha}\\
&\qquad -4{\rm tr}(\tilde{P}^{\alpha}D_{\beta}\tilde{P}^{\beta})+\frac{3}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}+4{\rm tr}(\tilde{P}^{\alpha}\tilde{P}_{\beta}\tilde{P}^{\beta})-\big[{\rm tr}\tilde{P}^{\alpha}+(D-4)\partial^{\alpha}\varphi\big](\tilde{F}^2+4\tilde{P}^2)\big]\\
&\qquad +D^{\alpha}\Big[e^{(D-4)\varphi}\tilde{e}_{\rm int}\Big(\frac{1}{2}\tilde{F}^2+2\tilde{P}^2\Big)\Big]\Big\}
\end{aligned}
\tag{A.20}
$$

and the Ricci tensor squared is given by

$$
\begin{aligned}
&\hat{e}e^{-4\varphi}\big(\tilde{R}_{AB}\big)^2\\
&\quad = ee^{(D-4)\varphi}\tilde{e}_{\rm int}\Big\{R_{\alpha\beta}\big[R^{\alpha\beta}-\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta}-2{\rm tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta})+{\rm tr}\tilde{P}^{\alpha}{\rm tr}\tilde{P}^{\beta}+2(D-4){\rm tr}\tilde{P}^{\alpha}\partial^{\beta}\varphi\big]\\
&\qquad -R\big[{\rm tr}(D_{\alpha}\tilde{P}^{\alpha})+({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_{\alpha}\partial^{\alpha}\varphi\big]+(D_{\alpha}\tilde{F}_{b\gamma}{}^{\alpha})\Big[\frac{1}{2}D_{\beta}\tilde{F}^{b\gamma\beta}+\tilde{P}_{\delta}{}^{bc}\tilde{F}_c{}^{\gamma\delta}\Big]\\
&\qquad +(D_{\alpha}\tilde{P}^{\alpha}{}_{bc})\Big[D_{\beta}\tilde{P}^{\beta bc}-\frac{1}{2}\tilde{F}^b{}_{\gamma\delta}\tilde{F}^{c\gamma\delta}\Big]+\frac{1}{4}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta}+\frac{1}{16}\tilde{F}_{c\alpha\beta}\tilde{F}_d{}^{\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}^{d\gamma\delta}\\
&\qquad +{\rm tr}(D_{\alpha}\tilde{P}^{\alpha})\Big[{\rm tr}(D_{\beta}\tilde{P}^{\beta})+\frac{1}{4}\tilde{F}^2+\tilde{P}^2+\frac{3}{2}({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_{\beta}\partial^{\beta}\varphi\Big]\\
&\qquad +\frac{1}{2}\tilde{F}^c{}_{\beta\gamma}\tilde{P}^{\gamma}{}_{ce}\tilde{P}_{\delta}{}^{ed}\tilde{F}_d{}^{\beta\delta}+\tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_{\beta}\Big[{\rm tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta})-\frac{1}{2}{\rm tr}\tilde{P}^{\alpha}{\rm tr}\tilde{P}^{\beta}-(D-4){\rm tr}\tilde{P}^{\alpha}\partial^{\beta}\varphi\Big]\\
&\qquad +\frac{1}{4}\tilde{F}^2\big[({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_{\alpha}\partial^{\alpha}\varphi\big]+\tilde{P}^2\big[({\rm tr}\tilde{P})^2+(D-4){\rm tr}\tilde{P}_{\alpha}\partial^{\alpha}\varphi\big]\\
&\qquad +{\rm tr}(\tilde{P}_{\alpha}\tilde{P}_{\beta})\big[{\rm tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta})-{\rm tr}\tilde{P}^{\alpha}{\rm tr}\tilde{P}^{\beta}-2(D-4){\rm tr}\tilde{P}^{\alpha}\partial^{\beta}\varphi\big]\\
&\qquad +\frac{1}{2}({\rm tr}\tilde{P})^2\big[({\rm tr}\tilde{P})^2+2(D-4){\rm tr}\tilde{P}_{\alpha}\partial^{\alpha}\varphi+(D-4)^2(\partial\varphi)^2+(D-4)\Box\varphi\big]\big\}\\
&\qquad +e\nabla_{\alpha}\big\{e^{(D-4)\varphi}\tilde{e}_{\rm int}\Big[\big(-2R^{\alpha\beta}+\eta^{\alpha\beta}R+\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta}+2{\rm tr}(\tilde{P}^{\alpha}\tilde{P}^{\beta})\\
&\qquad -(D-4)\partial^{\alpha}\varphi{\rm tr}\tilde{P}^{\beta}\big){\rm tr}\tilde{P}_{\beta}-\Big(\frac{1}{4}\tilde{F}^2+{\rm tr}(D_{\beta}\tilde{P}^{\beta})+{\rm tr}(\tilde{P}_{\beta}\tilde{P}^{\beta})+({\rm tr}\tilde{P})^2\Big){\rm tr}\tilde{P}^{\alpha}\Big]\\
&\qquad +\frac{1}{2}D^{\alpha}\big[e^{(D-4)\varphi}\tilde{e}_{\rm int}({\rm tr}\tilde{P})^2\big]\Big\}.
\end{aligned}
\tag{A.21}
$$

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Using also $\tilde{\Box}\varphi = \Box\varphi + \partial_\alpha\varphi \mathrm{tr}\tilde{P}^\alpha$ and $(\tilde{\partial}\varphi)^2 = (\partial\varphi)^2$ we have all the ingredients to extract the compactified Gauss-Bonnet Lagrangian, eq. (A.9). Notice that $\hat{e}(\hat{\nabla}_A\hat{X}^A) = \hat{\partial}_M(\hat{e}\hat{X}^M) = \partial_\mu(\hat{e}\hat{X}^\mu)$ holds after the compactification as well, implying that the total derivative terms in eq. (A.9) will still be total derivatives even after the compactification. Together with the result from the Weyl-rescaling, eq. (A.9), the complete result after the compactification is

$$
\begin{aligned}
\hat{e}\hat{R}^2_{\mathrm{GB}} = {} & \sqrt{|g|}\tilde{e}_{\mathrm{int}}e^{(D-4)\varphi}\Big\{[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2] - \frac{1}{2}[R_{\alpha\beta\gamma\delta}\tilde{F}_c{}^{\alpha\beta}\tilde{F}^{c\gamma\delta} + R\tilde{F}^2 \\
& -4R_{\alpha\beta}\tilde{F}_{c\delta}{}^{\alpha}\tilde{F}^{c\delta\beta}] + \frac{1}{8}\tilde{F}_{c\alpha\beta}\tilde{F}^c{}_{\gamma\delta}\tilde{F}_d{}^{\alpha\gamma}\tilde{F}^{d\beta\delta} - \frac{1}{2}\tilde{F}_{c\gamma\alpha}\tilde{F}^{c\gamma}{}_{\beta}\tilde{F}_{d\delta}{}^{\alpha}\tilde{F}^{d\delta\beta} + \frac{(D-n)}{16(D-n-2)}(\tilde{F}^2)^2 \\
& +2\tilde{F}^c{}_{\beta\gamma}\tilde{P}_{\delta ce}\tilde{P}^{\gamma ed}\tilde{F}_d{}^{\beta\delta} + \tilde{F}^2\left(\frac{1}{2}(\mathrm{tr}\tilde{P})^2 + (D-4)\mathrm{tr}\tilde{P}_\beta\partial^\beta\varphi + \frac{(D-4)^2}{2}(\partial\varphi)^2\right) \\
& -\frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}^{\alpha cd}\tilde{F}_d{}^{\beta\gamma}(\mathrm{tr}\tilde{P}_\alpha + (D-4)\partial_\alpha\varphi) - 2\tilde{F}_{c\beta\gamma}\tilde{P}^{\gamma cd}\tilde{F}_d{}^{\beta\alpha}(\mathrm{tr}\tilde{P}_\alpha + (D-4)\partial_\alpha\varphi) \\
& +2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^\alpha\tilde{P}^\beta) + 2\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - (\tilde{P}^2)^2 - 4(D-2)\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)\partial^\alpha\varphi\partial^\beta\varphi \\
& -4\mathrm{tr}(\tilde{P}_\alpha\tilde{P}^\alpha\tilde{P}^\beta)(\mathrm{tr}\tilde{P}_\beta + (D-4)\partial_\beta\varphi) - 4(D-4)(\mathrm{tr}\tilde{P})^2\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi \\
& +2\tilde{P}^2((\mathrm{tr}\tilde{P})^2 + 2(D-4)\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi + (D^2-7D+14)(\partial\varphi)^2) - (\mathrm{tr}\tilde{P})^2(\mathrm{tr}\tilde{P})^2 \\
& -2(D^2-7D+14)(\mathrm{tr}\tilde{P})^2(\partial\varphi)^2 - 4(D^2-8D+14)(\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi)^2 \\
& -4(D-4)(D^2-7D+11)\mathrm{tr}\tilde{P}_\alpha\partial^\alpha\varphi(\partial\varphi)^2 - (D-2)(D-3)(D-4)(D-5)(\partial\varphi)^2(\partial\varphi)^2 \\
& +\Big[R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R - \frac{1}{2}\tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_{\beta} + \frac{1}{8}\eta_{\alpha\beta}\tilde{F}^2 - \mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta) + \frac{1}{2}\eta_{\alpha\beta}\tilde{P}^2 + (D-2)\partial_\alpha\varphi\partial_\beta\varphi \\
& -\frac{(D-2)}{2}\eta_{\alpha\beta}(\partial\varphi)^2\Big](4\mathrm{tr}(\tilde{P}^\alpha\tilde{P}^\beta) - 4\mathrm{tr}\tilde{P}^\alpha\mathrm{tr}\tilde{P}^\beta - 8(D-4)\mathrm{tr}\tilde{P}^\alpha\partial^\beta\varphi \\
& -4(D-3)(D-4)\partial^\alpha\varphi\partial^\beta\varphi) \\
& +[D_\alpha\tilde{F}_{c\delta}{}^{\alpha} + \tilde{P}_{\alpha c}{}^{e}\tilde{F}_{e\delta}{}^{\alpha}](-2\tilde{F}_d{}^{\delta\gamma}\tilde{P}_\gamma{}^{cd} + 2\tilde{F}^{c\delta\gamma}\mathrm{tr}\tilde{P}_\gamma + 2(D-4)\tilde{F}^{c\delta\gamma}\partial_\gamma\varphi) \\
& +\Big[D_\alpha\tilde{P}^\alpha{}_{cd} - \frac{1}{4}\tilde{F}_{c\alpha\beta}\tilde{F}_d{}^{\alpha\beta} - \frac{1}{(D-2)}\delta_{cd}\mathrm{tr}(D_\alpha\tilde{P}^\alpha)\Big]\Big(- \frac{1}{2}\tilde{F}^c{}_{\gamma\delta}\tilde{F}^{d\gamma\delta} - 4\tilde{P}_{\gamma e}{}^{c}\tilde{P}^{\gamma ed} \\
& +4\tilde{P}^{\gamma cd}\mathrm{tr}\tilde{P}_\gamma + 4(D-4)\tilde{P}^{\gamma cd}\partial_\gamma\varphi\Big) \\
& +\frac{1}{(D-2)}\Big[\mathrm{tr}(D_\alpha\tilde{P}^\alpha) - \frac{(D-2)}{4(D-n-2)}\tilde{F}^2\Big]\Big(\frac{(D-3)}{2}\tilde{F}^2 + 2(D-4)\tilde{P}^2 - 2(D-4)(\mathrm{tr}\tilde{P})^2 \\
& -4(D-3)(D-4)\mathrm{tr}\tilde{P}_\beta\partial^\beta\varphi - 2(D-2)(D-3)(D-4)(\partial\varphi)^2\Big) \\
& +2(D-4)\Big[\Box\varphi + \frac{1}{4(D-n-2)}\tilde{F}^2\Big]\Big(\frac{1}{4}\tilde{F}^2 + \tilde{P}^2 - (\mathrm{tr}\tilde{P})^2 - 2(D-3)\mathrm{tr}\tilde{P}_\beta\partial^\beta\varphi \\
& -(D-2)(D-3)(\partial\varphi)^2\Big)\Big\} \\
& +\mathcal{L}_{TD},
\end{aligned}
\tag{A.22}
$$

where the last term, $\mathcal{L}_{TD}$, is a total derivative which is given explicitly by

$$
\begin{aligned}
\mathcal{L}_{TD} = \sqrt{|g|}D^{\alpha}\Big\{ & D_\alpha\Big[\tilde{e}_{\rm int}e^{(D-4)\varphi}\Big(\frac{1}{2}\tilde{F}^2+2\tilde{P}^2-2(\mathrm{tr}\tilde{P})^2\Big)\Big]+\tilde{e}_{\rm int}e^{(D-4)\varphi}\Big[\\
& +8\Big[R_{\alpha\beta}-\frac{1}{2}\eta_{\alpha\beta}R-\frac{1}{2}\tilde{F}_{c\delta\alpha}\tilde{F}^{c\delta}{}_{\beta}+\frac{1}{8}\eta_{\alpha\beta}\tilde{F}^2-\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta)+\frac{1}{2}\eta_{\alpha\beta}\tilde{P}^2+(D-2)\partial_\alpha\varphi\partial_\beta\varphi \\
& -\frac{(D-2)}{2}\eta_{\alpha\beta}(\partial\varphi)^2\Big]\mathrm{tr}\tilde{P}^{\beta}-4\Big[D_\beta\tilde{P}^{\beta cd}-\frac{1}{4}\tilde{F}^{c}{}_{\beta\gamma}\tilde{F}^{d\beta\gamma}-\frac{1}{(D-2)}\delta^{cd}\mathrm{tr}(D_\beta\tilde{P}^{\beta})\Big]\tilde{P}_{\alpha cd} \\
& -2[D_\gamma\tilde{F}^{c\delta\gamma}+\tilde{P}_{\gamma}{}^{ce}\tilde{F}_{e}{}^{\delta\gamma}]\tilde{F}_{c\delta\alpha}+4\frac{(D-3)}{(D-2)}\Big[\mathrm{tr}(D_\beta\tilde{P}^{\beta})-\frac{(D-2)}{4(D-n-2)}\tilde{F}^2\Big]\mathrm{tr}\tilde{P}_\alpha \\
& +\frac{1}{2}\tilde{F}_{c\beta\gamma}\tilde{P}_{\alpha}{}^{cd}\tilde{F}_{d}{}^{\beta\gamma}+2\tilde{F}_{c\beta\alpha}\tilde{P}_{\gamma}{}^{cd}\tilde{F}_{d}{}^{\beta\gamma}+\frac{(n-1)}{(D-n-2)}\tilde{F}^2\mathrm{tr}\tilde{P}_\alpha-(D-4)\tilde{F}^2\partial_\alpha\varphi \\
& +4\mathrm{tr}(\tilde{P}_\alpha\tilde{P}_\beta\tilde{P}^{\beta})+4((\mathrm{tr}\tilde{P})^2-\mathrm{tr}(\tilde{P}_\beta\tilde{P}^{\beta}))(\mathrm{tr}\tilde{P}_\alpha+(D-4)\partial_\alpha\varphi) \\
& +(D^2-9D+16)(4\partial_\alpha\varphi\,\mathrm{tr}\tilde{P}_\beta-2\mathrm{tr}\tilde{P}_\alpha\partial_\beta\varphi)\partial^{\beta}\varphi\Big]\Big\}. \qquad \text{(A.23)}
\end{aligned}
$$

All terms are thus grouped according to equations of motion and total derivatives. The first two square parenthesis in eq. (A.22) — containing terms involving only the Riemann tensor and $\tilde{F}$ — will vanish identically when compactifying to three dimensions.

Varying the compactified Einstein-Hilbert action, $\mathcal{L}_{\rm EH}=\hat{e}\hat{R}$, the tree-level equations of motion are found to be

$$
\begin{aligned}
0 &= R_{\alpha\beta}-\tilde{P}_{\alpha cd}\tilde{P}_{\beta}{}^{cd}-\frac{1}{2}\tilde{F}_{c\alpha\delta}\tilde{F}^{c}{}_{\beta}{}^{\delta}+\frac{1}{4(D-n-2)}\eta_{\alpha\beta}\tilde{F}^2+(D-2)\partial_\alpha\varphi\partial_\beta\varphi, \\
0 &= D_\gamma\tilde{F}_{a}{}^{\beta\gamma}+\tilde{P}_{\gamma ad}\tilde{F}^{d\beta\gamma}, \\
0 &= D_\gamma\tilde{P}^{\gamma}{}_{ab}-\frac{1}{4}\tilde{F}_{a\gamma\delta}\tilde{F}_{b}{}^{\gamma\delta}-\frac{1}{4(D-n-2)}\delta_{ab}\tilde{F}^2. \qquad \text{(A.24)}
\end{aligned}
$$

Notice that tracing the last equation in eq. (A.24), one finds $\Box\varphi+\frac{1}{4(D-n-2)}\tilde{F}^2=0$ for the dilatons. Except for the equations of motion, the fields will also obey the Bianchi identities

$$\nabla_{[a}\tilde{F}^{m}{}_{\beta\gamma]}=0, \qquad \text{(A.25)}$$

and the Maurer-Cartan equations

$$
\begin{aligned}
0 &= D_{[\alpha}\tilde{P}_{\beta]cd}, \\
0 &= \nabla_{[\alpha}Q_{\beta]cd}-Q_{[\alpha|c|}{}^{e}Q_{\beta]de}+\tilde{P}_{[\alpha|c|}{}^{e}\tilde{P}_{\beta]de}. \qquad \text{(A.26)}
\end{aligned}
$$

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